



Binomial theorem, functions and polynomials

IB Mathematics: Analysis and Approaches HL

50 original questions ■ Non-calculator test preparation

Learning goal	Build accuracy in exact algebra, graph interpretation and connected HL problems.
Syllabus link	SL 1.9; AHL 1.10 (binomial expansions); SL 2.1–2.11 and AHL 2.12–2.16; AHL 1.12–1.14 for complex roots.
Structure	15 short responses; 15 extended responses; 20 IB-style Paper 1 questions. Answer key and solution outlines begin on page 18.
Technology	No calculator throughout. Topic 2 graphing and equation skills are assessed by hand; technology-only operations are outside this pack.

Instructions. Show your working in a separate notebook. Give exact answers unless an approximation is requested. State excluded values and reject inadmissible roots. Label intercepts, asymptotes and key points on sketches. Use radians for arguments. Unless stated otherwise, functions have their largest real domains. Work through this pack over several sessions. Marks indicate the expected depth; these are original practice questions, not an official IB examination.

1 | Short response

Questions 1–15

1. (3 marks)

Expand $(2 - x)^5$ in descending powers of x . Hence state the coefficient of x^3 . [3]

2. (2 marks)

Find the coefficient of x^4 in the expansion of $(1 + 2x)^6$. [2]

3. (3 marks)

Expand $(1 - 2x)^{-1}$ up to and including the term in x^3 . State the interval of convergence. [3]



Short response

Questions 4–9

4. (3 marks)

Find the equation of the line through $(2, -1)$ perpendicular to $3x - 2y + 4 = 0$. Give your answer in the form $ax + by + c = 0$, with integer coefficients. [3]

5. (2 marks)

State the largest real domain and the range of $f(x) = \sqrt{7 - 2x}$. [2]

6. (3 marks)

The function f has domain $\{-2, -1, 0, 1, 2\}$ and is defined by the table.

x	-2	-1	0	1	2
$f(x)$	3	1	-1	1	3

(a) State $f(0)$ and solve $f(x) = 1$. [2]

(b) Explain why f does not have an inverse function on this domain. [1]

7. (3 marks)

Let $f(x) = 3x - 2$ and $g(x) = x^2 + 1$. Find $(f \circ g)(x)$ and $(g \circ f)(0)$. [3]

8. (3 marks)

Write $f(x) = 2x^2 - 12x + 13$ in vertex form. State its minimum value and the equation of its axis of symmetry. [3]

9. (2 marks)

Solve $x^2 - x - 6 \leq 0$. [2]



Short response

Questions 10–15

10. (4 marks)

For $f(x) = \frac{3x-1}{x+2}$, state both asymptotes and find the coordinates of the intercepts with the axes. [4]

11. (3 marks)

Solve $e^{2x} - 5e^x + 4 = 0$, giving exact answers. [3]

12. (3 marks)

A point (a, b) lies on $y = f(x)$.

(a) Write the coordinates of the corresponding point on $y = -2f(-3(x-1)) + 4$. [2]

(b) Hence find the image of the point $(6, -1)$. [1]

13. (3 marks)

Determine whether each function is odd, even or neither. Justify your answers algebraically.

(a) $f(x) = x^4 - 3x^2$. [1]

(b) $g(x) = \frac{x}{1+x^2}$. [2]

14. (2 marks)

Find the remainder when $P(x) = 2x^3 - 3x^2 + 4x - 7$ is divided by $x - 2$. [2]

15. (3 marks)

Solve $z^2 - 4z + 13 = 0$ over $\mathbb{C}$. State the sum and the product of the roots. [3]



2 | Extended response

Questions 16–30

16. (6 marks)

The coefficient of x^2 in $(1 + ax)^5$ is 160, where $a > 0$.

- (a) Find a . [3]
- (b) Hence find the coefficient of x^3 in $(1 - 2x)(1 + ax)^5$. [3]
-

17. (6 marks)

- (a) Expand $(1 + 3x)^{-2}$ up to and including the term in x^3 , stating the interval of convergence. [4]
- (b) Find the coefficient of x^3 in the expansion of $(1 + x)(1 + 3x)^{-2}$. [2]
-

18. (6 marks)

- (a) Write $\sqrt{4 + x}$ as $2(1 + x/4)^{1/2}$ and expand up to and including the term in x^3 . State an open interval on which the expansion is valid. [4]
- (b) Use these four terms to obtain an approximation to $\sqrt{4.4}$. Give your answer as a fraction. [2]
-

19. (7 marks)

The function $f(x) = x^2 - 6x + 5$ has domain $x \geq 3$.

- (a) Write f in vertex form and state its range. [2]
- (b) Find $f^{-1}(x)$, stating its domain. [3]
- (c) Verify that $f^{-1}(f(x)) = x$ for $x \geq 3$. [2]
-

20. (7 marks)

Let $f(x) = \sqrt{x - 1}$ and $g(x) = \frac{3}{x + 2}$, each with its largest real domain.

- (a) Find $(f \circ g)(x)$ and its domain. [4]
- (b) Find $(g \circ f)(x)$ and its domain. [3]
-

21. (6 marks)

Consider $x^2 - 2kx + k + 2 = 0$, where $k \in \mathbb{R}$.

- (a) Find the values of k for which the equation has two distinct real roots. [4]
- (b) Find the values of k for which it has a repeated root, and state the root in each case. [2]
-



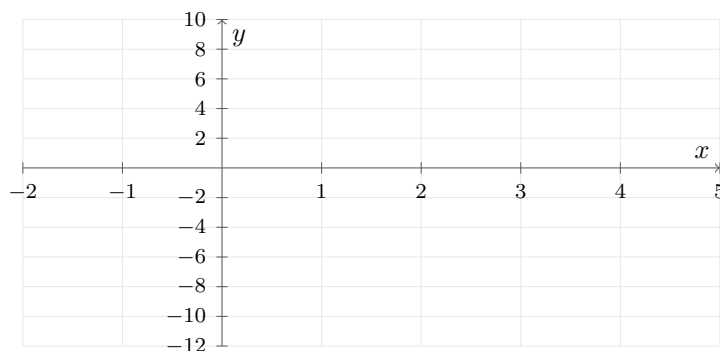
Extended response

Questions 22–23

22. (8 marks)

The function $f(x) = -2(x - 1)^2 + 8$ is defined for $-1 \leq x \leq 4$.

- (a) State the vertex and find all intercepts with the axes. [3]
(b) Sketch $y = f(x)$, marking its endpoints, and state its range. [3]
(c) Find the exact coordinates of the intersections of $y = f(x)$ and $y = 2$. [2]

**23.** (7 marks)

Let $f(x) = \frac{2x + 3}{x - 1}$.

- (a) Write $f(x)$ in the form $a + \frac{b}{x - 1}$. State its domain and range. [2]
(b) Find $f^{-1}(x)$, stating its domain and range. [3]
(c) Verify algebraically that $f(f^{-1}(x)) = x$ on the domain of f^{-1} . [2]



Extended response

Questions 24–27

24. (7 marks)

Solve each equation, giving exact answers and showing any necessary domain restrictions.

(a) $\log_2(x-1) + \log_2(x-3) = 3.$ [4]

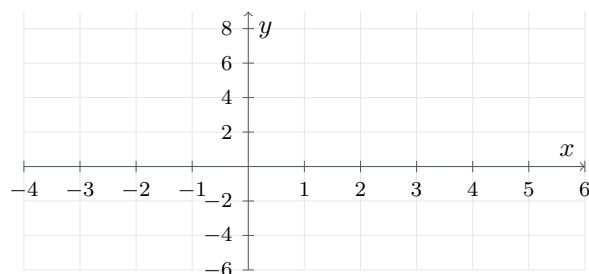
(b) $2^{2x} - 6 \cdot 2^x + 8 = 0.$ [3]

25. (7 marks)Let $P(x) = x^3 - 2x^2 - 5x + 6.$ (a) Show that $x - 1$ is a factor, and factorize $P(x)$ completely over $\mathbb{R}.$ [4](b) Hence solve $P(x) \geq 0.$ [3]**26.** (6 marks)A monic polynomial P of degree four has real coefficients. Its roots include $1 + 2i$, and -2 is a root of multiplicity two.

(a) State the remaining root. [1]

(b) Express $P(x)$ as a product of real factors, and then expand it. [3]

(c) State the sum and product of all four roots, counting multiplicity. [2]

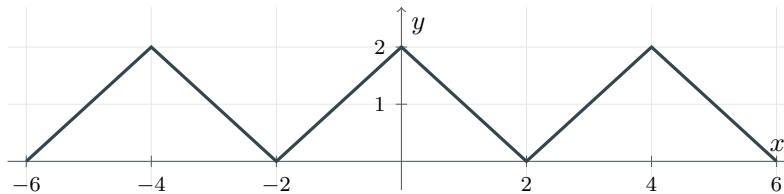
27. (8 marks)Let $f(x) = \frac{x^2 + 1}{x - 1}.$ (a) Use polynomial division to express $f(x)$ as a linear term plus a proper fraction. [2](b) State all asymptotes and find all intercepts with the axes. Sketch $y = f(x)$, showing these features. Exact coordinates of turning points are not required. [4](c) Show that $f(x) = 1$ has no real solution. [2]

Extended response

Questions 28–30

28. (7 marks)

The graph shows part of a periodic function f . On $[-2, 2]$, $f(x) = 2 - |x|$, and this pattern repeats every four units.



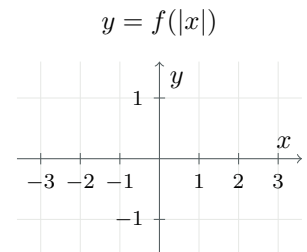
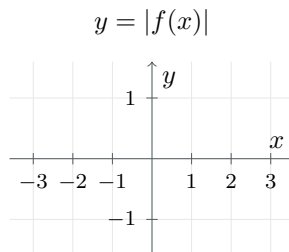
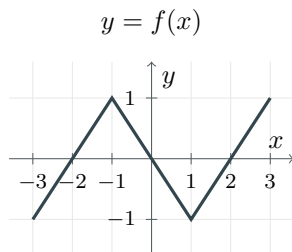
- (a) Find $f(5)$ and $f(-7)$. [2]
- (b) State the least positive period and whether f is odd, even or neither. [2]
- (c) Solve $f(x) = \frac{1}{2}$ for $-2 \leq x \leq 6$. [3]

29. (6 marks)

The graph of f consists of line segments joining $(-3, -1)$, $(-1, 1)$, $(1, -1)$ and $(3, 1)$, in that order. On the two empty grids, sketch:

- (a) $y = |f(x)|$; [3]
- (b) $y = f(|x|)$. [3]

Label all intercepts with the x -axis on both sketches.



30. (6 marks)

Solve each of the following, showing your reasoning.

- (a) $|2x - 3| = x + 1$. [3]
- (b) $|x - 2| < 2x + 1$. [3]



3 | IB-style Paper 1

Questions 31–50

31. (8 marks)

For a positive integer $n \geq 2$ and a positive constant k ,

$$(1 + kx)^n = 1 + 12x + 60x^2 + \dots$$

- (a) Show that $nk = 12$ and $n(n-1)k^2 = 120$, and hence find n and k . [4]
- (b) Find the coefficient of x^3 in $(1 + kx)^n$. [2]
- (c) Hence find the coefficient of x^3 in $(1 - x)(1 + kx)^n$. [2]

32. (8 marks)

Consider the expansion of $\left(2x^2 - \frac{1}{x}\right)^6$, where $x \neq 0$.

- (a) Find the term independent of x . [3]
- (b) Find the coefficient of x^3 . [3]
- (c) Explain why there is no term in x . [2]

33. (11 marks)

Let $f(x) = (1 - 3x)^{-1/2}$.

- (a) Expand $f(x)$ up to and including the term in x^3 . [4]
- (b) State the interval of convergence. [1]
- (c) Find constants a and b such that the coefficients of x and x^2 in $(1 + ax + bx^2)f(x)$ are both zero. [4]
- (d) For these values of a and b , find the coefficient of x^3 in this product. [2]



IB-style Paper 1

Questions 34–36

34. (9 marks)

- (a) Expand $(1 + x)^{1/3}$ up to and including the term in x^3 . State an open interval on which the expansion is valid. [4]
- (b) Use only the terms up to and including x^2 to approximate $\sqrt[3]{9}$. Give your answer as a fraction. [3]
- (c) For the substitution used in part (b), state the first omitted term in the approximation to $\sqrt[3]{9}$. Does this term increase or decrease the approximation? [2]

35. (10 marks)

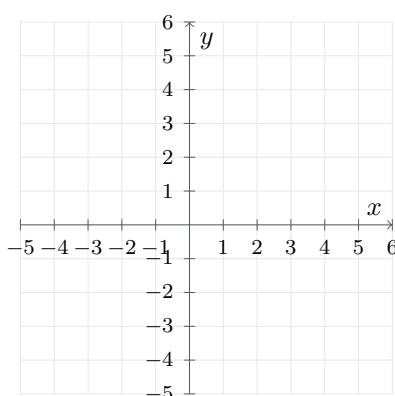
The line $y = 2x + k$ is tangent to the parabola $y = x^2 - 4x + 1$.

- (a) Find k using the discriminant. [3]
- (b) Find the point of tangency. [2]
- (c) Find the equation of the line through this point perpendicular to the tangent. [2]
- (d) The function $f(x) = x^2 - 4x + 1$ is restricted to $x \geq 2$. Find $f^{-1}(x)$ and state its domain. [3]

36. (10 marks)

Let $f(x) = \frac{x+3}{x-1}$, with its largest real domain.

- (a) Write $f(x)$ in the form $a + \frac{b}{x-1}$. State its asymptotes and find its intercepts. [3]
- (b) Find $f^{-1}(x)$ and hence show that f is self-inverse. [3]
- (c) Find the coordinates of the intersections of $y = f(x)$ and $y = x$. [2]
- (d) Sketch $y = f(x)$ and $y = x$, showing the symmetry associated with the inverse. [2]





IB-style Paper 1

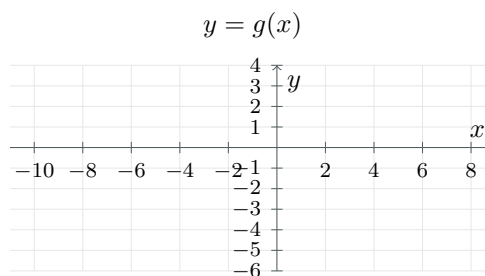
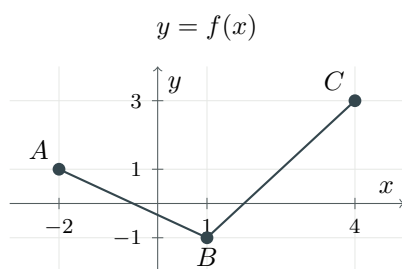
Questions 37–38

37. (9 marks)

The graph of f consists of the line segments AB and BC , where $A = (-2, 1)$, $B = (1, -1)$ and $C = (4, 3)$. Let

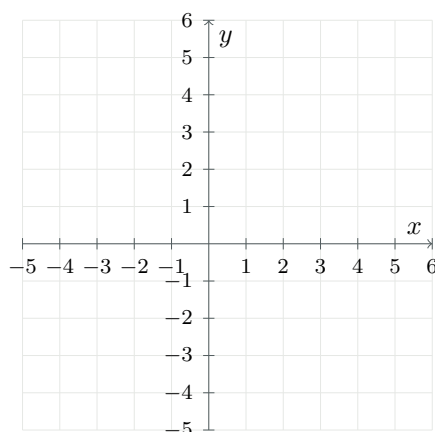
$$g(x) = -2f\left(\frac{x+4}{3}\right) + 1.$$

- (a) Describe one valid ordered sequence of transformations taking $y = f(x)$ to $y = g(x)$. [3]
(b) State the domain and range of g . [3]
(c) Find the images of A , B and C , and sketch $y = g(x)$. [3]

**38.** (10 marks)

Let $f(x) = 4 - 2^{x+1}$, for $x \in \mathbb{R}$.

- (a) State the range and horizontal asymptote of f , and find its intercepts. [3]
(b) Find $f^{-1}(x)$ and state its domain. [3]
(c) Solve $f(x) \geq -4$. [2]
(d) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, labelling both asymptotes. [2]





IB-style Paper 1

Questions 39–40

39. (9 marks)The mass M grams of a substance after t hours is modelled by

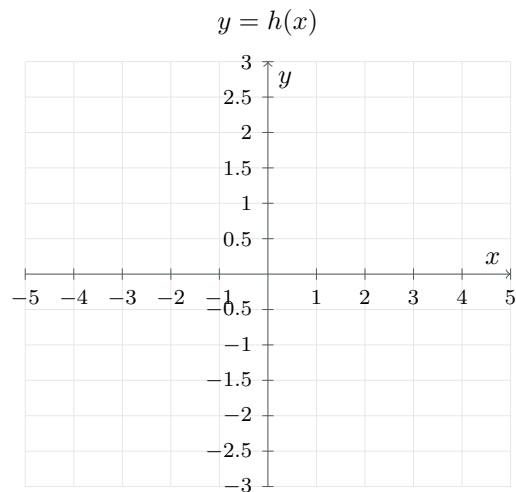
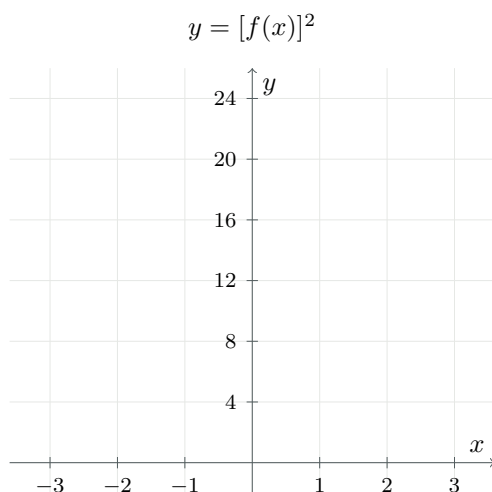
$$M(t) = A 2^{-kt}, \quad t \geq 0, \quad A, k > 0.$$

t (hours)	0	6
$M(t)$ (grams)	80	10

- (a) Find A and k . [3]
- (b) Find the half-life: the time taken for the mass to halve. [2]
- (c) Find when the mass reaches 5 grams. [2]
- (d) State the range of M for this model. Explain whether the mass ever becomes zero at a finite time. [2]

40. (10 marks)Let $f(x) = x^2 - 4$.

- (a) Sketch $y = [f(x)]^2$, labelling all intercepts and local maximum and minimum points. You need not use differentiation. [3]
- (b) For $h(x) = \frac{1}{f(x)}$, state its domain, parity (odd/even/neither), asymptotes and intercepts. Sketch $y = h(x)$ on the second grid. [4]
- (c) Solve $h(x) \geq \frac{1}{5}$ algebraically. [3]





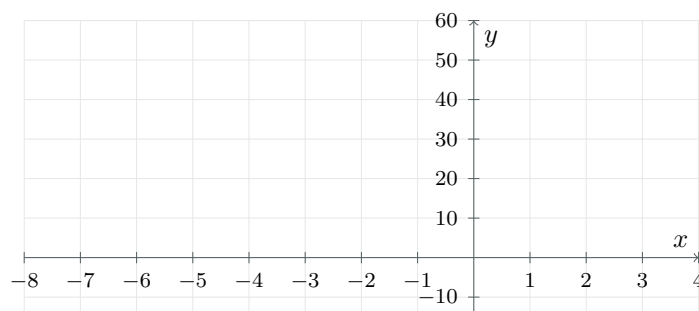
IB-style Paper 1

Questions 41–42

41. (12 marks)

Let $P(x) = x^3 + ax^2 + bx + 6$. The polynomial is divisible by $x - 1$, and the remainder on division by $x - 2$ is 8.

- (a) Find a and b . [4]
- (b) Factorize $P(x)$ completely, stating the multiplicity of each root. [3]
- (c) Sketch $y = P(x)$, showing its intercepts and its behaviour at each root and at both ends. Exact coordinates of turning points are not required. [3]
- (d) Hence solve $P(x) < 0$. [2]

**42.** (10 marks)

The roots of $2x^3 - 3x^2 - 8x + 12 = 0$ are α, β, γ .

- (a) State $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$. [2]
- (b) Factorize the polynomial by grouping and find its three roots. [3]
- (c) Find a monic cubic polynomial whose roots are $\alpha + 1, \beta + 1, \gamma + 1$. Give your answer in expanded form. [3]
- (d) Find $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. [2]



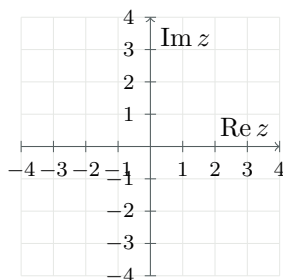
IB-style Paper 1

Questions 43–44

43. (10 marks)

The polynomial $P(z) = z^3 + az^2 + bz + 20$ has real coefficients and a root $1 + 3i$.

- (a) State another root and explain why it must be a root. [2]
- (b) Find the third root, and hence find a and b . [4]
- (c) Factorize $P(z)$ completely into linear factors over $\mathbb{C}$. [2]
- (d) Represent the three roots on an Argand diagram. [2]

**44.** (10 marks)

A monic quartic polynomial P has real coefficients and roots $2 + i$ and $-1 + 2i$.

- (a) State its other two roots. [2]
- (b) Find $P(x)$ in expanded form. [4]
- (c) State the sum and product of all four roots. [2]
- (d) Prove that $P(x) > 0$ for every real x . [2]

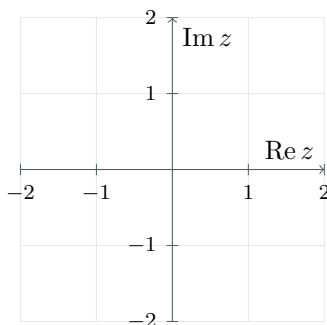
**45.** (9 marks)

- (a) Use De Moivre's theorem to find all roots of $z^3 = 8$ in Cartesian form. [4]
- (b) Hence factorize $z^3 - 8$ into a linear factor and a quadratic factor with real coefficients. [2]
- (c) Find the monic polynomial of least degree whose roots are the squares of the three roots found in part (a). [3]
-

46. (10 marks)

Consider $z^4 + 4 = 0$.

- (a) By writing $z^4 + 4 = (z^2 + 2)^2 - (2z)^2$, factorize $z^4 + 4$ into two real quadratic factors. [3]
- (b) Find all four roots in Cartesian form. [3]
- (c) State the modulus and principal argument of each root, using $-\pi < \arg z \leq \pi$. [2]
- (d) Plot the roots on an Argand diagram and state the shape formed when adjacent roots are joined. [2]





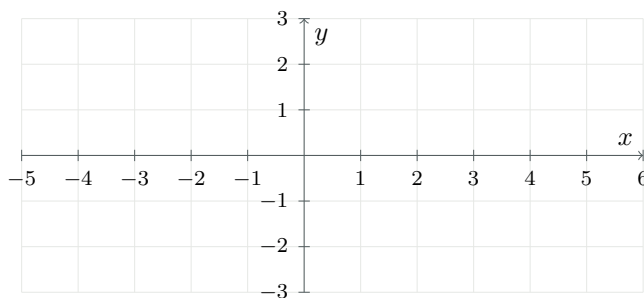
IB-style Paper 1

Questions 47–48

47. (10 marks)

Let $f(x) = \frac{x+1}{x^2-x-2}$, with its largest real domain.

- (a) State the domain of f and simplify its formula, retaining the original restrictions. [2]
- (b) State all asymptotes and the coordinates of any missing point (a hole). Sketch the graph, marking the hole with an open circle. [3]
- (c) State the range of f . Explain why $y = f(x)$ does not intersect $y = -\frac{1}{3}$. [2]
- (d) Solve $f(x) \geq 1$. [3]

**48.** (11 marks)

The curves $y = x^3 - 3x$ and $y = kx$ are considered for real k .

- (a) Show that the x -coordinates of their intersections satisfy $x(x^2 - k - 3) = 0$. [3]
- (b) Determine the number of distinct real intersection points for each possible value of k . [3]
- (c) When $k = 1$, solve $x^3 - 3x \geq x$. [3]
- (d) When $k = -2$, find the coordinates of every intersection point. [2]



IB-style Paper 1

Questions 49–50

49. (10 marks)

Let $f(x) = \frac{x^2 + 4}{2x}$, with domain $x > 0$.

- (a) Show that $f(x) - 2 = \frac{(x-2)^2}{2x}$. Hence find the minimum value of f and state its range. [3]
 - (b) Restrict f to $x \geq 2$. Find its inverse function and state the domain of the inverse. [3]
 - (c) Instead restrict f to $0 < x \leq 2$. Find the inverse function for this restriction. [2]
 - (d) Show that the two inverse expressions have product 4. Interpret this result in terms of the positive solutions of $f(x) = c$, for $c > 2$. [2]
-

50. (12 marks)

Let $f(x) = (1+x)^{-2}$, with domain $x > -1$.

- (a) Expand $f(x)$ up to and including x^3 , and state the interval of convergence. [3]
- (b) Let $P_3(x) = 1 - 2x + 3x^2 - 4x^3$. Show that $(1+x)^2 P_3(x) = 1 - 5x^4 - 4x^5$. [3]
- (c) Hence find an exact expression for $f(x) - P_3(x)$, and prove that $P_3(x)$ underestimates $f(x)$ for $0 < x < 1$. [3]
- (d) Solve $f(x) = 4$. Find $f^{-1}(x)$, stating its domain. [3]



Revision coverage map

Use after marking

This table helps you return to a weak area. All Topic 2 sections are represented through their non-calculator content; calculator procedures are deliberately excluded.

Syllabus	Skills	Questions
SL 1.9	Positive-integer binomial expansions; coefficients and general terms	1, 2, 16, 31, 32
AHL 1.10	Negative and rational powers; convergence; approximations; products of expansions	3, 17, 18, 33, 34, 50
SL 2.1	Lines, gradients, perpendicularity	4, 35
SL 2.2	Domain, range, notation and mathematical models	5, 6, 20, 22, 37, 39, 47, 49
SL 2.3–2.4	Sketching, reading graphs, features and intersections	6, 22, 28, 36–38, 40, 41, 47
SL 2.5	Composites, inverses and identity compositions	7, 19, 20, 23, 36, 38, 50
SL 2.6–2.7	Quadratic forms, roots, inequalities and discriminants	8, 9, 19, 21, 22, 35
SL 2.8	Reciprocal and linear-over-linear rational functions	10, 23, 36
SL 2.9–2.10	Exponential/logarithmic functions; exact equations and models	11, 24, 38, 39, 48, 50
SL 2.11	Composite transformations	12, 37, 38
AHL 2.12	Factor/remainder theorems, multiplicities, polynomial roots, sums and products	14, 15, 25, 26, 41–46, 48, 50
AHL 2.13	Rational functions, oblique asymptotes and excluded inputs	27, 40, 47, 49
AHL 2.14	Parity, periodicity, restricted inverses and self-inverse functions	13, 19, 28, 35, 36, 49, 50
AHL 2.15	Polynomial and rational inequalities; comparing functions	25, 38, 40, 41, 47, 48
AHL 2.16	$ f(x) $, $f(x)$, $1/f(x)$, $f(ax + b)$, $[f(x)]^2$; modulus equations/inequalities	12, 29, 30, 37, 40
AHL 1.12–1.14	Complex roots, conjugate pairs, Argand diagrams, modulus/argument and De Moivre	15, 26, 43–46

After marking

For each error, identify whether it came from algebra, a missing restriction, interpretation or presentation. Redo the question without the answer key, then try another question from the same row above.



Answer key and solution outlines

Exact answers throughout

Check the method as well as the final result. Alternative correct methods are valid. For sketches, the labelled features and overall behaviour matter more than plotting many points.

Section 1 | Short response

1. Using $\binom{5}{0}, \dots, \binom{5}{5} = 1, 5, 10, 10, 5, 1$,

$$(2-x)^5 = -x^5 + 10x^4 - 40x^3 + 80x^2 - 80x + 32.$$

The coefficient of x^3 is $\boxed{-40}$.

2. The coefficient is $\binom{6}{4}2^4 = 15 \times 16 = \boxed{240}$.

3. $(1-2x)^{-1} = \boxed{1+2x+4x^2+8x^3+\dots}$, valid for $|2x| < 1$, so $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$.

4. The given gradient is $3/2$, so the perpendicular gradient is $-2/3$. Thus $y+1 = -\frac{2}{3}(x-2)$, giving $\boxed{2x+3y-1=0}$.

5. $7-2x \geq 0$ gives domain $\boxed{(-\infty, 7/2]}$; range $\boxed{[0, \infty)}$.

6.

- (a) $\boxed{f(0) = -1}$; $\boxed{x = -1, 1}$.

- (b) f is not one-to-one: for example, $f(-1) = f(1) = 1$.

7. $(f \circ g)(x) = 3(x^2+1) - 2 = \boxed{3x^2+1}$. Also $f(0) = -2$, so $(g \circ f)(0) = g(-2) = \boxed{5}$.

8. $f(x) = 2[(x-3)^2 - 9] + 13 = \boxed{2(x-3)^2 - 5}$. Minimum value $\boxed{-5}$; axis $\boxed{x=3}$.

9. $(x-3)(x+2) \leq 0$. The quadratic is non-positive between its roots: $\boxed{-2 \leq x \leq 3}$.

10. Asymptotes: $\boxed{x=-2}$ and $\boxed{y=3}$. Intercepts: $\boxed{(1/3, 0)}$ and $\boxed{(0, -1/2)}$.

11. Let $u = e^x > 0$. Then $(u-1)(u-4) = 0$, so $e^x = 1$ or 4 . Hence $\boxed{x=0 \text{ or } x=\ln 4}$.

12.

- (a) Set $-3(x-1) = a$: the point maps to $\boxed{(1-a/3, 4-2b)}$.

- (b) $\boxed{(-1, 6)}$.

13.

- (a) $f(-x) = x^4 - 3x^2 = f(x)$, so $\boxed{\text{even}}$.

- (b) $g(-x) = -x/(1+x^2) = -g(x)$, so $\boxed{\text{odd}}$.

14. By the remainder theorem, the remainder is $P(2) = 16 - 12 + 8 - 7 = \boxed{5}$.

15. $(z-2)^2 = -9$, so $\boxed{z = 2 \pm 3i}$. Sum $\boxed{4}$; product $(2+3i)(2-3i) = 4+9 = \boxed{13}$.

Section 2 | Extended response

16.

(a) $(\frac{5}{2})a^2 = 160$, so $10a^2 = 160$. Since $a > 0$, $a = 4$.

(b) The relevant coefficients in $(1 + 4x)^5$ are 160 and 640. The required coefficient is $640 - 2(160) = 320$.

17.

(a) Using $1 + nu + \frac{n(n-1)}{2}u^2 + \frac{n(n-1)(n-2)}{6}u^3 + \dots$,

$$(1 + 3x)^{-2} = 1 - 6x + 27x^2 - 108x^3 + \dots, \quad |x| < \frac{1}{3}.$$

(b) The coefficient is $-108 + 27 = -81$.

18.

(a) $2[1 + \frac{1}{2}\frac{x}{4} - \frac{1}{8}(\frac{x}{4})^2 + \frac{1}{16}(\frac{x}{4})^3 + \dots]$ gives $2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \dots$, valid on $-4 < x < 4$.

(b) Put $x = 2/5$: $\sqrt{4.4} \approx 2 + \frac{1}{10} - \frac{1}{400} + \frac{1}{8000} = \frac{16781}{8000}$.

19.

(a) $f(x) = (x - 3)^2 - 4$; range $[-4, \infty)$.

(b) $y + 4 = (x - 3)^2$ and $x - 3 \geq 0$, so $f^{-1}(x) = 3 + \sqrt{x + 4}$, domain $[-4, \infty)$.

(c) $f^{-1}(f(x)) = 3 + \sqrt{(x - 3)^2} = 3 + |x - 3| = x$ because $x \geq 3$.

20.

(a) $(f \circ g)(x) = \sqrt{\frac{1-x}{x+2}}$. Require $x \neq -2$ and $(1-x)/(x+2) \geq 0$; a sign check at -2 and 1 gives domain $(-2, 1]$.

(b) $(g \circ f)(x) = \frac{3}{\sqrt{x-1} + 2}$. Require $x \geq 1$; the denominator is then positive. Domain $[1, \infty)$.

21.

(a) $\Delta = (-2k)^2 - 4(k+2) = 4(k-2)(k+1) > 0$. Hence $k < -1$ or $k > 2$.

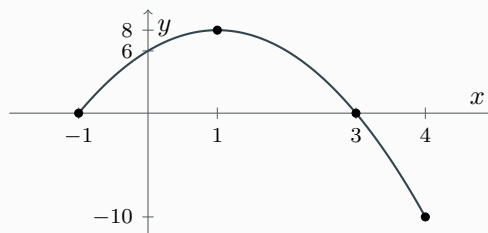
(b) $\Delta = 0$: $k = -1$ gives $(x+1)^2 = 0$, root $x = -1$; $k = 2$ gives $(x-2)^2 = 0$, root $x = 2$.

22.

(a) Vertex $(1, 8)$; intercepts $(-1, 0), (3, 0), (0, 6)$.

(b) Endpoints $(-1, 0)$ and $(4, -10)$; the downward-opening parabola has range $[-10, 8]$.

(c) $-2(x-1)^2 + 8 = 2$ gives $(x-1)^2 = 3$. Both values are in the domain, so the intersections are $(1 - \sqrt{3}, 2), (1 + \sqrt{3}, 2)$.



Section 2 | Extended response (continued)

23.

(a) $f(x) = 2 + \frac{5}{x-1}$; domain $\mathbb{R} \setminus \{1\}$, range $\mathbb{R} \setminus \{2\}$.

(b) From $y(x-1) = 2x+3$, $f^{-1}(x) = \frac{x+3}{x-2}$. Its domain is $\mathbb{R} \setminus \{2\}$ and range is $\mathbb{R} \setminus \{1\}$.

(c) $f(f^{-1}(x)) = \frac{(2x+6+3x-6)/(x-2)}{(x+3-x+2)/(x-2)} = \frac{5x/(x-2)}{5/(x-2)} = x$, for $x \neq 2$.

24.

(a) Require $x > 3$. Then $(x-1)(x-3) = 2^3 = 8$, so $(x-5)(x+1) = 0$. Reject $x = -1$: $x = 5$.

(b) With $u = 2^x > 0$, $(u-2)(u-4) = 0$. Thus $x = 1$ or $x = 2$.

25.

(a) $P(1) = 1 - 2 - 5 + 6 = 0$. Division gives $P(x) = (x-1)(x^2 - x - 6) = (x-1)(x-3)(x+2)$.

(b) The signs across the simple roots $-2, 1, 3$ are $-, +, -, +$. Hence $[-2, 1] \cup [3, \infty)$.

26.

(a) $1 - 2i$, by the conjugate-root theorem.

(b) $P(x) = [(x-1)^2 + 4](x+2)^2 = x^4 + 2x^3 + x^2 + 12x + 20$.

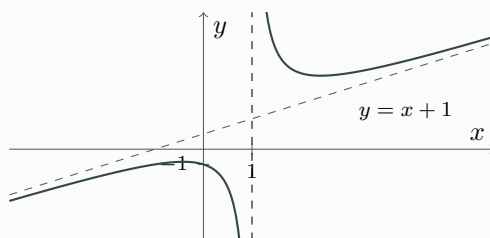
(c) Sum -2 ; product 20 .

27.

(a) $x^2 + 1 = (x-1)(x+1) + 2$, so $f(x) = x + 1 + \frac{2}{x-1}$.

(b) Asymptotes $x = 1$, $y = x + 1$; intercept $(0, -1)$. There are no x -intercepts since $x^2 + 1 > 0$.

(c) $f(x) = 1$ would give $x^2 - x + 2 = 0$, with discriminant $1 - 8 = -7 < 0$.



28.

(a) $f(5) = f(1) = 1$; $f(-7) = f(1) = 1$.

(b) Least positive period 4 ; even (symmetry in the y -axis).

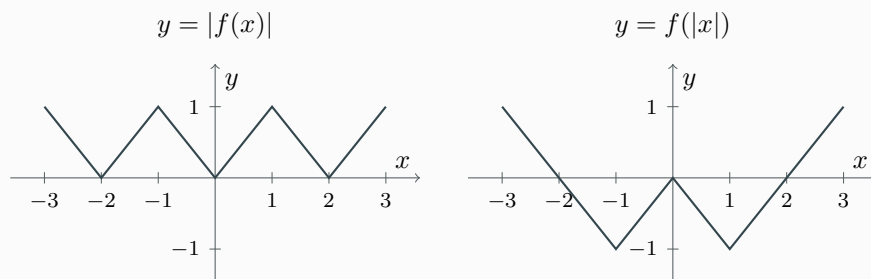
(c) On $[-2, 2]$, $2 - |x| = 1/2$ gives $x = \pm 3/2$. Add 4 to obtain the next pair: $x = -\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{11}{2}$.

Section 2 | Extended response (continued)

29.

(a) Reflect the parts below the x -axis upwards. Join $(-3, 1), (-2, 0), (-1, 1), (0, 0), (1, 1), (2, 0), (3, 1)$.

(b) Keep the part with $x \geq 0$, then reflect it in the y -axis. Join $(-3, 1), (-1, -1), (0, 0), (1, -1), (3, 1)$. In both cases the x -intercepts are $(-2, 0), (0, 0), (2, 0)$.



30.

(a) Require $x + 1 \geq 0$. The cases $2x - 3 = x + 1$ and $3 - 2x = x + 1$ give $x = 4$ or $x = 2/3$; both satisfy the original equation.

(b) Require $2x + 1 > 0$ and $-(2x + 1) < x - 2 < 2x + 1$. The inequalities give $x > 1/3$ and $x > -3$; combining all restrictions yields $x > 1/3$.



Section 3 | IB-style Paper 1

31.

(a) Comparing coefficients gives $nk = 12$ and $\frac{n(n-1)}{2}k^2 = 60$. Divide the second stated relation by $(nk)^2 = 144$: $(n-1)/n = 120/144 = 5/6$. Hence $n = 6, k = 2$.

(b) $\binom{6}{3}2^3 = 20(8) = 160$.

(c) $160 - 60 = 100$.

32. The general term is $T_{r+1} = \binom{6}{r}(2x^2)^{6-r}(-x^{-1})^r = \binom{6}{r}2^{6-r}(-1)^r x^{12-3r}$, $0 \leq r \leq 6$.

(a) $12 - 3r = 0$ gives $r = 4$, so the constant term is $\binom{6}{4}2^2 = 60$.

(b) $12 - 3r = 3$ gives $r = 3$, so the coefficient is $-\binom{6}{3}2^3 = -160$.

(c) $12 - 3r = 1$ requires $r = 11/3$, which is not an integer term index.

33.

(a) Substituting $n = -1/2$ and $u = -3x$ in the binomial series gives $f(x) = 1 + \frac{3}{2}x + \frac{27}{8}x^2 + \frac{135}{16}x^3 + \dots$.

(b) $|x| < 1/3$.

(c) $a + 3/2 = 0$ gives $a = -3/2$. Then $b + \frac{3}{2}a + \frac{27}{8} = 0$ gives $b = -9/8$.

(d) $\frac{135}{16} - \frac{3}{2} \cdot \frac{27}{8} - \frac{9}{8} \cdot \frac{3}{2} = \frac{135-81-27}{16} = \frac{27}{16}$.

34.

(a) $(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} + \dots$, valid on $-1 < x < 1$.

(b) $\sqrt[3]{9} = 2(1 + 1/8)^{1/3}$, so $\sqrt[3]{9} \approx 2(1 + \frac{1}{24} - \frac{1}{576}) = \frac{599}{288}$.

(c) The first omitted term is $2 \cdot \frac{5}{81}(\frac{1}{8})^3 = \frac{5}{20736}$. It is positive, so it **increases** the approximation.

35.

(a) Intersections satisfy $x^2 - 6x + 1 - k = 0$. Tangency requires $\Delta = 36 - 4(1 - k) = 0$, giving $k = -8$.

(b) The repeated root is $x = 3$, and $y = 6 - 8 = -2$: $(3, -2)$.

(c) The perpendicular gradient is $-1/2$: $y + 2 = -\frac{1}{2}(x - 3)$, or $y = -\frac{1}{2}x - \frac{1}{2}$.

(d) $f(x) = (x - 2)^2 - 3$, with $x - 2 \geq 0$. Thus $f^{-1}(x) = 2 + \sqrt{x + 3}$, domain $[-3, \infty)$.

Section 3 | IB-style Paper 1 (continued)

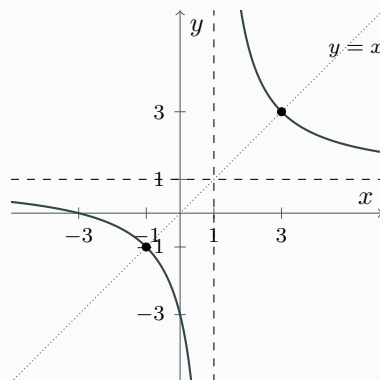
36.

(a) $f(x) = 1 + \frac{4}{x-1}$; asymptotes $x = 1, y = 1$; intercepts $(-3, 0), (0, -3)$.

(b) $y(x-1) = x+3$ gives $x = (y+3)/(y-1)$. Thus $f^{-1}(x) = (x+3)/(x-1) = f(x)$, both with domain $\mathbb{R} \setminus \{1\}$.

(c) $x+3 = x(x-1)$ gives $(x-3)(x+1) = 0$: $(-1, -1), (3, 3)$.

(d) The two branches of the hyperbola are invariant under reflection in $y = x$.

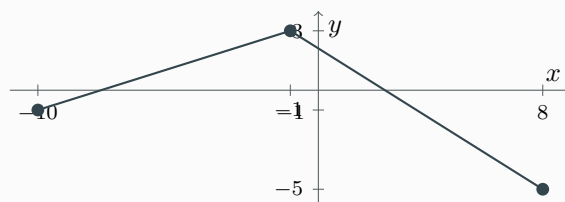


37.

(a) One sequence: horizontal stretch, scale factor 3; translation 4 units left; vertical stretch, scale factor 2; reflection in the x -axis; translation 1 unit up.

(b) A point (u, v) maps to $(3u-4, 1-2v)$. Domain $[-2, 4]$ maps to $[-10, 8]$; range $[-1, 3]$ maps to $[-5, 3]$.

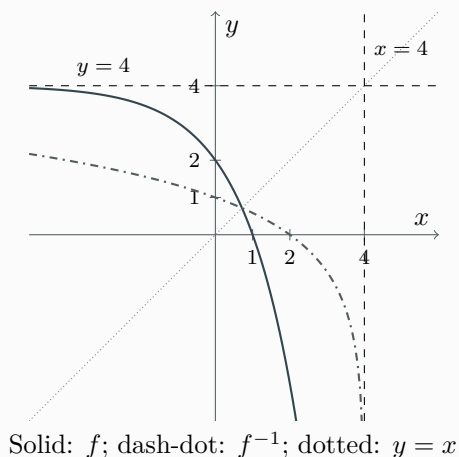
(c) $A' = (-10, -1), B' = (-1, 3), C' = (8, -5)$. Join these three points with line segments, including both endpoints.



Section 3 | IB-style Paper 1 (continued)

38.

- (a) Since $2^{x+1} > 0$, range $(-\infty, 4)$; asymptote $y = 4$. Intercepts $(1, 0), (0, 2)$.
- (b) $2^{x+1} = 4 - y$ gives $f^{-1}(x) = \log_2(4 - x) - 1$, domain $(-\infty, 4)$; its range is $\mathbb{R}$.
- (c) $2^{x+1} \leq 8 = 2^3$, hence $x \leq 2$.
- (d) Reflect f in $y = x$. The inverse has asymptote $x = 4$ and intercepts $(2, 0), (0, 1)$.

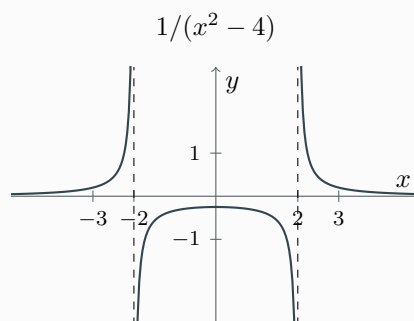
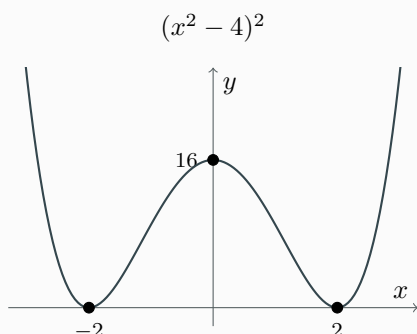


39.

- (a) $M(0) = A = 80$. Then $10 = 80 \cdot 2^{-6k}$ gives $2^{-6k} = 1/8 = 2^{-3}$, so $A = 80, k = 1/2$.
- (b) $2^{-t/2} = 1/2$ gives 2 hours .
- (c) $80 \cdot 2^{-t/2} = 5$ gives $2^{-t/2} = 1/16 = 2^{-4}$, so $t = 8 \text{ hours}$.
- (d) Range $0 < M \leq 80$ grams. $M(t) > 0$ for every finite t , and $M(t) \rightarrow 0$ as $t \rightarrow \infty$.

40.

- (a) $[f(x)]^2 = (x^2 - 4)^2 \geq 0$. Minima and x -intercepts: $(-2, 0), (2, 0)$. Local maximum and y -intercept: $(0, 16)$. The graph is even and tends to $+\infty$ at both ends.
- (b) Domain $\mathbb{R} \setminus \{-2, 2\}$; h is even. Asymptotes $x = -2, x = 2, y = 0$; y -intercept $(0, -1/4)$; no x -intercepts. It is positive for $|x| > 2$ and negative for $|x| < 2$.
- (c) $h(x) - \frac{1}{5} = \frac{9-x^2}{5(x^2-4)} \geq 0$. The critical values are $-3, -2, 2, 3$; exclude ± 2 . A sign table gives $[-3, -2) \cup (2, 3]$.



Section 3 | IB-style Paper 1 (continued)

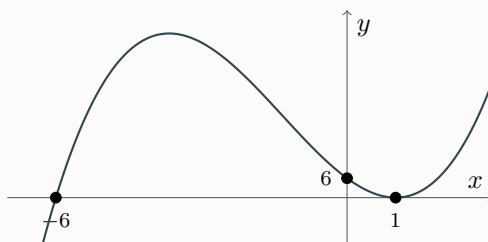
41.

(a) $P(1) = 0$ gives $a + b = -7$; $P(2) = 8$ gives $2a + b = -3$. Subtracting yields $a = 4, b = -11$.

(b) $P(x) = x^3 + 4x^2 - 11x + 6 = (x - 1)(x^2 + 5x - 6) = (x - 1)^2(x + 6)$. The root 1 has multiplicity two; -6 has multiplicity one.

(c) Crosses at $(-6, 0)$, touches at $(1, 0)$, passes through $(0, 6)$; left end down, right end up. The local maximum need not be calculated.

(d) $(x - 1)^2 \geq 0$ and $x + 6 < 0$ only when $x < -6$: $x < -6$.



42.

(a) Sum $3/2$; product -6 .

(b) $x^2(2x - 3) - 4(2x - 3) = (2x - 3)(x - 2)(x + 2)$. Roots $3/2, 2, -2$.

(c) New roots $5/2, 3, -1$ give $(x - \frac{5}{2})(x - 3)(x + 1) = x^3 - \frac{9}{2}x^2 + 2x + \frac{15}{2}$.

(d) $\frac{2}{3} + \frac{1}{2} - \frac{1}{2} = \frac{2}{3}$.

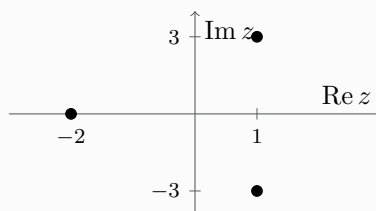
43.

(a) $1 - 3i$, since non-real roots of a real-coefficient polynomial occur in conjugate pairs.

(b) The product of all roots is -20 . Since $(1 + 3i)(1 - 3i) = 10$, the third root is -2 . Then $(z + 2)(z^2 - 2z + 10) = z^3 + 6z + 20$, so $a = 0, b = 6$.

(c) $P(z) = (z + 2)(z - 1 - 3i)(z - 1 + 3i)$.

(d) Plot $(-2, 0), (1, 3), (1, -3)$ with real and imaginary axes.



44.

(a) $2 - i, -1 - 2i$.

(b) $P(x) = [(x - 2)^2 + 1][(x + 1)^2 + 4] = (x^2 - 4x + 5)(x^2 + 2x + 5) = x^4 - 2x^3 + 2x^2 - 10x + 25$.

(c) Sum 2 ; product 25 .

(d) For real x , $(x - 2)^2 + 1 \geq 1 > 0$ and $(x + 1)^2 + 4 \geq 4 > 0$. Their product is strictly positive.

45.

(a) $8 = 8 \operatorname{cis}(2k\pi)$ gives $z = 2 \operatorname{cis}(2k\pi/3)$, $k = 0, 1, 2$. Thus $z = 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$.

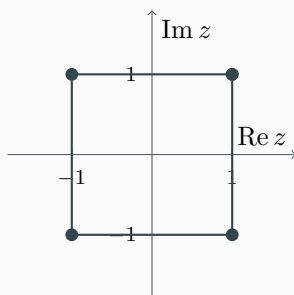
(b) The conjugate pair has sum -2 and product 4 : $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$.

(c) The squared roots are $4, -2 - 2i\sqrt{3}, -2 + 2i\sqrt{3}$. They are three distinct roots of $w^3 = 64$, so the required polynomial is $w^3 - 64$ (equivalently $x^3 - 64$).

Section 3 | IB-style Paper 1 (continued)

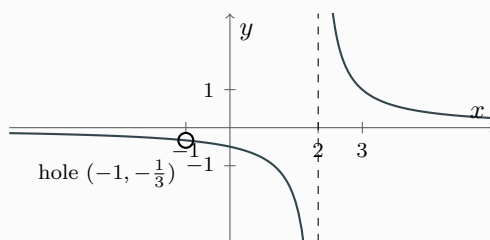
46.

- (a) Difference of squares gives $(z^2 - 2z + 2)(z^2 + 2z + 2)$.
- (b) $(z - 1)^2 = -1$ or $(z + 1)^2 = -1$, giving $1 + i, 1 - i, -1 + i, -1 - i$.
- (c) Each modulus is $\sqrt{2}$. The principal arguments, respectively, are $\pi/4, -\pi/4, 3\pi/4, -3\pi/4$.
- (d) The points $(\pm 1, \pm 1)$ form a square.



47.

- (a) $x^2 - x - 2 = (x - 2)(x + 1)$, so $f(x) = \frac{1}{x - 2}, x \neq -1, 2$.
- (b) Asymptotes $x = 2, y = 0$; hole $(-1, -1/3)$. The graph passes through $(0, -1/2)$ and has no x -intercept.
- (c) Range $\mathbb{R} \setminus \{0, -1/3\}$. The equation $1/(x - 2) = -1/3$ gives only $x = -1$, which is excluded.
- (d) $\frac{1}{x-2} - 1 = \frac{3-x}{x-2} \geq 0$ gives $2 < x \leq 3$; the excluded value -1 is not in this interval.



48.

- (a) Equate the expressions for y : $x^3 - 3x = kx \iff x^3 - (k + 3)x = 0 \iff x(x^2 - k - 3) = 0$.
- (b) $x = 0$ always. Two additional roots $\pm\sqrt{k+3}$ are real and distinct from 0 iff $k > -3$. Thus 3 points if $k > -3$; 1 point if $k \leq -3$. At $k = -3$, the equation is $x^3 = 0$, so the repeated root still gives just one point.
- (c) $x(x - 2)(x + 2) \geq 0$; signs $-, +, -, +$ give $[-2, 0] \cup [2, \infty)$.
- (d) $x(x^2 - 1) = 0$, so $x = -1, 0, 1$. With $y = -2x$, the points are $(-1, 2), (0, 0), (1, -2)$.

49.

- (a) $f(x) - 2 = (x^2 - 4x + 4)/(2x) = (x - 2)^2/(2x) \geq 0$ because $x > 0$. Equality occurs at $x = 2$. Minimum 2 ; range $[2, \infty)$. Every value $y \geq 2$ occurs, since $x = y + \sqrt{y^2 - 4} > 0$ solves $f(x) = y$.
- (b) $y = (x^2 + 4)/(2x)$ gives $x^2 - 2yx + 4 = 0$, so $x = y \pm \sqrt{y^2 - 4}$. For the branch $x \geq 2$, choose the plus sign: $f^{-1}(x) = x + \sqrt{x^2 - 4}$, domain $[2, \infty)$.
- (c) For $0 < x \leq 2$, $f^{-1}(x) = x - \sqrt{x^2 - 4}$, domain $[2, \infty)$.
- (d) $(x + \sqrt{x^2 - 4})(x - \sqrt{x^2 - 4}) = x^2 - (x^2 - 4) = 4$. For $c > 2$, the two distinct positive inputs giving output c multiply to 4.



Section 3 | IB-style Paper 1 (continued)

50.

(a) $f(x) = 1 - 2x + 3x^2 - 4x^3 + \dots$; $-1 < x < 1$.

(b) Multiplying $P_3(x)$ successively by $1, 2x, x^2$ and collecting terms gives

$$\begin{aligned}(1 + 2x + x^2)P_3(x) &= 1 + (-2 + 2)x + (3 - 4 + 1)x^2 \\ &\quad + (-4 + 6 - 2)x^3 + (-8 + 3)x^4 - 4x^5 \\ &= 1 - 5x^4 - 4x^5.\end{aligned}$$

(c) $f(x) - P_3(x) = \frac{x^4(5 + 4x)}{(1 + x)^2}$. For $0 < x < 1$, every factor shown is positive, so $f(x) > P_3(x)$.

(d) $(1 + x)^2 = 1/4$ gives $x = -1/2$ or $-3/2$; the domain excludes $-3/2$. Thus $x = -1/2$. Since $1 + x > 0$, $y = (1 + x)^{-2}$ implies $f^{-1}(x) = \frac{1}{\sqrt{x}} - 1$, domain $x > 0$.