



# Number and algebra revision

IB Mathematics AA SL & HL

300 questions | Sequences, finance, logarithms and binomial expansions

|                      |                                                                                                                                         |
|----------------------|-----------------------------------------------------------------------------------------------------------------------------------------|
| <b>Learning goal</b> | Strengthen core techniques, connect representations and apply methods in multi-part problems.                                           |
| <b>Syllabus link</b> | SL 1.2, 1.3, 1.4, 1.5, 1.7, 1.8, 1.9; AHL 1.10 negative and rational binomial expansions only.                                          |
| <b>Structure</b>     | Q1–240: shared SL/HL revision. Q241–300: HL-exclusive extension. Each topic progresses from short to extended and exam-style responses. |
| <b>Technology</b>    | Every question is marked NC or  GDC. There are 254 NC and 46 GDC questions. Use a calculator only for GDC questions.                    |

**Instructions.** Work on separate paper. Show the formula or equation before substituting. Give exact answers unless a question asks for an approximation. For GDC questions, round money to the nearest cent and other numerical answers to 3 significant figures unless stated otherwise. Keep unrounded values in intermediate work.

Throughout,  $\log x = \log_{10} x$  and  $\ln x = \log_e x$ . In sequences,  $u_1$  is the first term and  $S_n$  is the sum of the first  $n$  terms. In finance, no fees, taxes, deposits or withdrawals apply unless stated. On a TI-Nspire CX, use Scratchpad Calculate or the finance solver as appropriate.

**Reading the labels.** NC means non-calculator: manageable exact arithmetic or a symbolic answer is expected. GDC means calculator practice. In exam-style sections, the NC questions follow Paper 1 conventions and the GDC questions follow Paper 2 conventions. All questions are original.

## Find your way

| Code                             | Topic                                                              | Questions | Page |
|----------------------------------|--------------------------------------------------------------------|-----------|------|
| 1.2                              | Arithmetic sequences and series                                    | 1–35      | 2    |
| 1.3                              | Geometric sequences and finite series                              | 36–70     | 6    |
| 1.4                              | Financial mathematics                                              | 71–105    | 10   |
| 1.5                              | Integer exponents and introductory logarithms                      | 106–135   | 15   |
| 1.7                              | Rational exponents, logarithm laws and equations                   | 136–175   | 18   |
| 1.8                              | Infinite geometric series                                          | 176–205   | 22   |
| 1.9                              | Binomial theorem: nonnegative integer powers                       | 206–240   | 26   |
| AHL 1.10                         | HL exclusive: binomial series with negative and rational exponents | 241–300   | 30   |
| Answer key and solution outlines |                                                                    | 1–300     | 37   |

**Revision route.** Start with the short-response questions in a topic. Use extended responses to practise combining ideas, then attempt its exam-style set. SL students finish at Q240. HL students complete the shared material before the extension.



## 1.2 | Arithmetic sequences and series

## Short response

1. NC An arithmetic sequence has  $u_1 = 4$  and  $d = 3$ . Find  $u_{12}$ .  

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2. NC An arithmetic sequence has  $u_1 = 25$  and  $d = -2$ . Find  $u_{10}$ .  

---
3. NC An arithmetic sequence has  $u_1 = -7$  and  $d = 5$ . Find  $u_9$ .  

---
4. NC An arithmetic sequence has  $u_1 = \frac{1}{2}$  and  $d = \frac{3}{2}$ . Find  $u_8$ .  

---
5. NC Find a formula for  $u_n$  for  $9, 5, 1, -3, \dots$   

---
6. NC In an arithmetic sequence,  $u_4 = 17$  and  $u_9 = 37$ . Find  $d$  and  $u_1$ .  

---
7. NC Find the sum of the first 20 terms of  $3, 7, 11, \dots$   

---
8. NC Evaluate  $\sum_{k=1}^{10} (2k + 1)$ .  

---
9. NC Write  $6 + 10 + 14 + \dots + 42$  using sigma notation.  

---
10. NC Find the first negative term of the arithmetic sequence  $31, 27, 23, \dots$

**Extended response**

**11. NC** The plotted points belong to an arithmetic sequence.



- (a) Read off  $u_1$  and  $d$ .
  - (b) Find a formula for  $u_n$ .
  - (c) Find  $u_{20}$ .
- 

**12. NC** Complete the arithmetic-sequence table and find  $u_{15}$ .

| $n$   | 1 | 2                    | 3  | 4                    | 5                    |
|-------|---|----------------------|----|----------------------|----------------------|
| $u_n$ | 7 | <input type="text"/> | 17 | <input type="text"/> | <input type="text"/> |

---

**13. NC** A theatre has 18 seats in row 1 and 3 more seats in each successive row.

- (a) Find the number of seats in row 12.
  - (b) Find the total in the first 12 rows.
- 

**14. NC** A student saves EUR 8 in week 1 and increases the weekly amount by EUR 2.

- (a) Find the amount saved in week 15.
  - (b) Find the first week by which the total saved exceeds EUR 200.
- 

**15. NC** Three consecutive terms of an arithmetic sequence are  $x + 1$ ,  $3x - 2$ ,  $4x + 2$ . Find  $x$  and the three terms.

---

**16. NC** Insert four arithmetic means between 3 and 28.

---

**17. NC** The partial sums of a sequence are  $S_n = 2n^2 + 3n$ .

- (a) Find  $u_1$ .
  - (b) Find  $u_n$ .
  - (c) Show that the sequence is arithmetic.
- 

**18. NC** Find the sum of the integers from 40 to 90 inclusive.



**19. GDC** Approximate an arithmetic model for the measurements in the table, using the first and last measurements.

| $n$   | 1    | 2    | 3    | 4    | 5    |
|-------|------|------|------|------|------|
| $u_n$ | 14.1 | 17.0 | 20.2 | 22.8 | 25.9 |

- (a) Find a model for  $u_n$ .
- (b) Predict  $u_8$ .
- (c) Explain one limitation.

**20. NC** An arithmetic sequence has  $u_5 = 14$  and  $S_{10} = 155$ . Find the first term and common difference.

**Exam-style**

**21. NC**

[Maximum mark: 5]

An arithmetic sequence has  $u_3 = 11$  and  $u_8 = 31$ .

- (a) Find the first term and common difference. [3]
- (b) Find  $S_{20}$ . [2]

**22. NC**

[Maximum mark: 6]

A staircase uses 5 tiles in its first row and 2 more tiles in each subsequent row.

- (a) Write down  $u_n$ . [1]
- (b) Show that  $S_n = n(n + 4)$ . [2]
- (c) Find the number of rows if 221 tiles are used. [3]

**23. NC**

[Maximum mark: 5]

The first term of an arithmetic sequence is 42 and the common difference is  $-3$ .

- (a) Find the number of positive terms. [3]
- (b) Find their sum. [2]

**24. NC**

[Maximum mark: 6]

An arithmetic sequence has  $S_{10} = 170$  and  $S_{20} = 640$ .

- (a) Find  $a = u_1$  and  $d$ . [4]
- (b) Find  $u_{30}$ . [2]

**25. NC**

[Maximum mark: 6]

The terms  $u_2, u_5, u_9$  of an arithmetic sequence are  $p, 2p, 28$ .

- (a) Find  $p$ . [4]
- (b) Find the first term and common difference. [2]

**26. NC***[Maximum mark: 5]*Let  $u_n = 7 - 2n$ .

- (a) Write down  $u_1$  and  $d$ . [2]
- (b) Evaluate  $\sum_{n=4}^{12} u_n$ . [3]

**27. NC***[Maximum mark: 5]*

A company raises an annual salary of EUR 30000 by EUR 1500 each year. Year 1 pays EUR 30000.

- (a) Find the salary in year 8. [2]
- (b) Find the total salary paid over eight years. [3]

**28. NC***[Maximum mark: 6]*

A runner covers 2 km on day 1 and increases the daily distance by 0.5 km.

- (a) Find an expression for the total distance after  $n$  days. [3]
- (b) Find the first day by which the total exceeds 100 km. [3]

**29. NC***[Maximum mark: 5]*For an arithmetic sequence,  $S_n = 3n^2 - 2n$ .

- (a) Find the first term and common difference. [3]
- (b) Determine whether 73 is a term of the sequence. [2]

**30. NC***[Maximum mark: 5]*An arithmetic sequence starts  $-12, -7, -2, \dots$ 

- (a) Find  $S_n$ . [2]
- (b) Find the least positive integer  $n$  for which  $S_n > 0$ . [3]

**31. NC***[Maximum mark: 5]*The sum of the first  $n$  positive odd integers is considered.

- (a) Show that the sum is  $n^2$ . [2]
- (b) Find  $1 + 3 + \dots + 39$ . [2]
- (c) Find how many initial odd integers have sum 625. [1]

**32. NC***[Maximum mark: 5]*An arithmetic sequence has  $u_p = 17$  and common difference 4.

- (a) Find  $u_{p+6}$ . [2]
- (b) Find  $u_p + u_{p+1} + \dots + u_{p+6}$ . [3]

**33. NC***[Maximum mark: 5]*

A model predicts daily ticket sales of  $u_n = 80 + 6(n - 1)$  for  $1 \leq n \leq 20$ .

- (a) Find the predicted sales on day 20. [2]
- (b) Find the total predicted sales. [2]
- (c) State a reason the model may fail beyond day 20. [1]

**34. NC***[Maximum mark: 5]*

The arithmetic sequences  $a_n = 4n + 1$  and  $b_n = 7n - 14$  are compared.

- (a) Find the index at which their terms are equal. [3]
- (b) Find the sum of the terms of  $a_n$  up to that index. [2]

**35. NC***[Maximum mark: 6]*

An arithmetic sequence has 15 terms, total 180, and last term 26.

- (a) Find its first term. [2]
- (b) Find its common difference. [2]
- (c) Find its middle term and compare it with the mean of all 15 terms. [2]

### 1.3 | Geometric sequences and finite series

#### Short response

**36. NC** A geometric sequence has  $u_1 = 3$  and  $r = 2$ . Find  $u_7$ .

**37. NC** A geometric sequence has  $u_1 = 64$  and  $r = \frac{1}{2}$ . Find  $u_6$ .

**38. NC** A geometric sequence has  $u_1 = 2$  and  $r = -3$ . Find  $u_5$ .

**39. NC** A geometric sequence has  $u_1 = 81$  and  $r = \frac{1}{3}$ . Find  $u_5$ .

**40. NC** Find  $r$  and  $u_n$  for  $12, -6, 3, -\frac{3}{2}, \dots$

**41. NC** Find  $S_6$  for  $2, 6, 18, \dots$

**42. NC** Find  $S_5$  for  $48, 24, 12, \dots$



43. NC Evaluate  $\sum_{k=0}^5 2^k$ .

44. NC Write  $5 + 15 + 45 + \dots + 1215$  in sigma notation.

45. NC Find  $S_{11}$  for the geometric sequence  $4, -4, 4, -4, \dots$

**Extended response**

46. NC A geometric sequence has  $u_2 = 12$  and  $u_5 = 96$ .

(a) Find  $a$  and  $r$ .

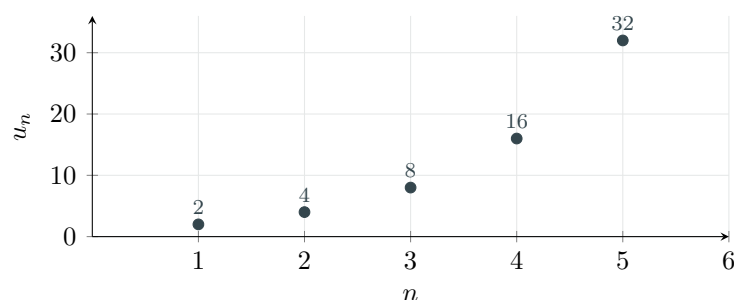
(b) Find  $S_5$ .

47. NC A geometric sequence has  $u_1 = 5$  and  $u_3 = 45$ .

(a) Find all possible common ratios.

(b) Find the corresponding values of  $u_4$ .

48. NC The plotted points form a geometric sequence.



(a) Find the first term and common ratio.

(b) Predict  $u_7$ .

49. NC Complete the table and find  $u_n$ .

| $n$   | 1 | 2                    | 3  | 4                    | 5   |
|-------|---|----------------------|----|----------------------|-----|
| $u_n$ | 3 | <input type="text"/> | 27 | <input type="text"/> | 243 |

50. GDC A population model is  $P_n = 200(1.15)^n$ , where  $n$  counts years after observation begins.

(a) State the initial population and annual percentage increase.

(b) Predict the population after six years.



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**51. GDC** A radioactive sample retains 80% of its mass each day. Its initial mass is 50 g.

- (a) Write a model for its mass after  $n$  days.  
(b) Find its mass after five days.
- 

**52. NC** Insert two positive geometric means between 4 and 108.

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**53. NC** For  $a = 7$  and  $r = 1$ , find  $u_{20}$  and  $S_{20}$ . Explain why the usual quotient formula for  $S_n$  cannot be used directly.

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**54. NC** A sequence is defined by  $u_n = 3 \cdot 2^{n-1}$ . Find the sum from  $u_4$  to  $u_8$  inclusive.

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**55. NC** A geometric sequence has partial sums  $S_n = 5(2^n - 1)$ . Find  $u_n$ ,  $a$  and  $r$ .

**Exam-style**

**56. NC**

[Maximum mark: 6]

A geometric sequence begins 6, 18, 54, ...

- (a) Find  $u_n$ . [1]  
(b) Find the index of the term 1458. [3]  
(c) Find the sum up to that term. [2]
- 

**57. NC**

[Maximum mark: 5]

The second and fifth terms of a geometric sequence are 18 and 486.

- (a) Find the first term and common ratio. [3]  
(b) Find the sum of the first four terms. [2]
- 

**58. NC**

[Maximum mark: 6]

The first three terms of a geometric sequence are  $x$ ,  $x + 6$ ,  $x + 18$ , with  $x > 0$ .

- (a) Find  $x$ . [3]  
(b) Find  $S_6$ . [3]
- 

**59. NC**

[Maximum mark: 5]

A geometric sequence has  $a = 96$  and  $r = \frac{1}{2}$ .

- (a) Find the index of the term 3. [3]  
(b) Find the sum up to and including that term. [2]
-

**60. NC***[Maximum mark: 5]*

A geometric sequence has  $u_1 = 2$  and  $u_3 = 18$ .

- (a) Find all possible ratios. [2]
- (b) Find the corresponding values of  $S_4$ . [3]

**61. NC***[Maximum mark: 5]*

A video is shared with 3 new people in round 1. In each later round, the number of new viewers triples.

- (a) Find the number of new viewers in round 6. [2]
- (b) Find the total number of new viewers in the first six rounds. [2]
- (c) State one limitation. [1]

**62. NC***[Maximum mark: 5]*

Let  $u_n = 8(-\frac{1}{2})^{n-1}$ .

- (a) Find  $u_4$  and  $u_5$ . [2]
- (b) Find  $S_6$ . [3]

**63.  GDC***[Maximum mark: 5]*

A culture initially contains 500 cells and increases by 18% each hour.

- (a) Write a model for the number after  $t$  hours. [2]
- (b) Find the first whole hour at which the model exceeds 2000 cells. [3]

**64. NC***[Maximum mark: 4]*

A geometric sequence has  $S_3 = 21$  and  $r = 2$ .

- (a) Find the first term. [2]
- (b) Find  $S_6$ . [2]

**65. NC***[Maximum mark: 4]*

The geometric sequence  $u_n = 4 \cdot 3^{n-1}$  is considered.

- (a) Find  $u_3$ . [1]
- (b) Find  $\sum_{n=3}^8 u_n$ . [3]

**66. NC***[Maximum mark: 5]*

A machine produces 160 units in its first hour. Production in each later hour is 75% of the previous hour.

- (a) Find the modelled production in hour 4. [2]
- (b) Find the modelled total for four hours. [3]

**67. NC***[Maximum mark: 5]*

An arithmetic sequence starts 2, 5, 8, ... and a geometric sequence starts 2, 4, 8, ...

- (a) Find each fourth term. [2]
- (b) Find each sum of four terms. [2]
- (c) Explain why matching first and third terms does not imply identical sequences. [1]

**68. NC***[Maximum mark: 6]*

For a geometric sequence with nonzero terms,  $u_3 = 20$  and  $u_7 = 320$ .

- (a) Find all real values of  $r$ . [2]
- (b) Find  $u_1$ . [2]
- (c) Find all possible values of  $u_6$ . [2]

**69. NC***[Maximum mark: 5]*

A geometric sequence has  $a = 5$ ,  $r = 2$ .

- (a) Find  $S_n$ . [2]
- (b) Find the least  $n$  for which  $S_n > 300$ . [3]

**70. NC***[Maximum mark: 5]*

The sum of the first  $n$  terms is  $S_n = 12(1 - 3^{-n})$ .

- (a) Find  $u_1$ . [1]
- (b) Find  $u_n$ . [3]
- (c) State the common ratio. [1]

## 1.4 | Financial mathematics

### Short response

**71. NC** EUR 2000 earns simple interest at 5% per year for three years. Find the interest and final amount.

**72. NC** EUR 1000 earns 10% compound interest annually for two years. Find its value.

**73. NC** A machine worth EUR 800 depreciates by 25% annually. Find its value after two years.

**74. NC** A nominal annual rate is 6%, compounded monthly. State the monthly rate and number of periods in five years.



**75. NC** Write a formula for EUR 5000 invested at a nominal annual rate of 4%, compounded quarterly, for  $t$  years.

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**76. NC** An investment grows from EUR 400 to EUR 484 over two years with annual compounding. Find the annual rate.

---

**77. NC** An item costs EUR 240 today. Annual inflation is 25%. Find its expected price in two years.

---

**78. NC** An investment is worth EUR 1210 in two years. Inflation is 10% per year. Find its value in today's euros.

---

**79. GDC** EUR 3500 is invested at a nominal annual rate of 4.2%, compounded monthly, for 6 years. Find the final value to the nearest cent.

---

**80. GDC** EUR 7200 is invested at a nominal annual rate of 3.6%, compounded quarterly, for 5 years. Find the final value to the nearest cent.

---

### Extended response

**81. GDC** EUR 2000 earns 5% interest annually. Compare simple and compound interest over three years.



Interest is added at the end of each year; no deposits or withdrawals.

- (a) Find the simple-interest value.
  - (b) Find the compound-interest value.
  - (c) Find the difference.
- 

**82. GDC** A car costs EUR 24000 and depreciates by 18% each year.

- (a) Find its value after five years.
  - (b) Write its value after  $n$  years.
- 

**83. GDC** Find the initial deposit needed to reach EUR 12000 in eight years at 4.5% interest compounded annually.

---

**84. GDC** EUR 5000 grows to EUR 6800 in six years. Interest compounds annually. Find the annual percentage rate to 3 significant figures.

---

**85. GDC** EUR 4000 earns 3.8% annual compound interest. Find the first whole year at which it exceeds EUR 6000.

---



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**86. GDC** A bank quotes 6% nominal annual interest compounded monthly. Find the effective annual rate to 3 significant figures.

---

**87. GDC** An investment earns 5% annually while inflation is 2% annually.

- (a) Find the annual real growth rate.
  - (b) Find the real value after four years of an initial EUR 8000.
- 

**88. GDC** Account A pays 4.8% compounded annually. Account B pays a nominal 4.7% compounded monthly.

- (a) Find each effective annual rate to 5 significant figures.
  - (b) State which has the greater one-year return, ignoring fees.
- 

**89. GDC** A machine is worth EUR 15000 initially and EUR 6144 after four years. Find its annual depreciation rate.

---

**90. GDC** A deposit of EUR 3000 earns a nominal 4% annual rate compounded half-yearly. Set up the finance solver for four years with no payments.

### Exam-style

**91. GDC**

[Maximum mark: 5]

EUR 6000 is invested at a nominal annual rate of 3.6%, compounded monthly.

- (a) Write its value after  $n$  years. [2]
  - (b) Find its value after seven years. [2]
  - (c) Find the total interest earned. [1]
- 

**92. GDC**

[Maximum mark: 5]

A van costs EUR 28000 and loses 16% of its value annually.

- (a) Find its value after six years. [2]
  - (b) Find the first whole year its value falls below EUR 10000. [3]
- 

**93. GDC**

[Maximum mark: 5]

EUR 9000 grows to EUR 11000 in five years under annual compounding.

- (a) Find the annual percentage rate. [3]
- (b) Find its value after ten years, retaining the unrounded rate. [2]

**94. 🧮 GDC***[Maximum mark: 5]*

An investment of EUR 10000 earns 4% annually. Inflation is 2.5% annually.

- (a) Find its nominal value after eight years. [2]
- (b) Find its value in today's euros. [2]
- (c) Comment on the change in purchasing power. [1]

**95. 🧮 GDC***[Maximum mark: 5]*

Account A offers 5% compounded annually. Account B offers 4.9% nominal annually, compounded quarterly. EUR 7000 is invested for six years.

- (a) Find the final value in each account. [4]
- (b) State which account is better if fees are equal. [1]

**96. NC***[Maximum mark: 5]*

A computer depreciates by 30% in year 1 and by 20% in year 2. Its original value is EUR 2000.

- (a) Find its value after two years. [2]
- (b) Find the total percentage depreciation. [2]
- (c) Explain why the loss is not 50%. [1]

**97. 🧮 GDC***[Maximum mark: 6]*

EUR 2500 is invested at a nominal annual rate of 4.2%, compounded monthly.

- (a) Find the effective annual rate. [3]
- (b) Find the first whole month at which the value exceeds EUR 5000. [3]

**98. 🧮 GDC***[Maximum mark: 5]*

An item costs EUR 80. Its price increases by 2.8% annually.

- (a) Find its expected price after 12 years. [2]
- (b) Find the first whole year its price exceeds EUR 120. [3]

**99. 🧮 GDC***[Maximum mark: 5]*

A fund must be worth EUR 50000 in ten years. Interest is 3.2% nominal annually, compounded quarterly.

- (a) Find the initial deposit required. [3]
- (b) Find the deposit required if the nominal annual rate were 4% instead. [2]

**100. 🧮 GDC***[Maximum mark: 5]*

The values of an asset are given below. Assume a constant annual percentage depreciation.

| Year        | 0     | 1    | 2    | 3 |
|-------------|-------|------|------|---|
| Value / EUR | 12000 | 9600 | 7680 | □ |

- (a) Find the depreciation rate. [2]
- (b) Find the missing value. [1]
- (c) Predict its value after five years. [2]

**101. 🧮 GDC***[Maximum mark: 5]*

A nominal annual rate of  $r\%$ , compounded half-yearly, has an effective annual rate of 6.09%.

- (a) Find  $r$ . [3]
- (b) Find the value of EUR 1000 after two years. [2]

**102. 🧮 GDC***[Maximum mark: 5]*

An investment earns 3% per year but inflation is 4% per year.

- (a) Find the annual real growth rate. [2]
- (b) Find the real value after five years of an initial EUR 6000. [2]
- (c) Interpret the result. [1]

**103. 🧮 GDC***[Maximum mark: 5]*

EUR 5000 earns 4% annually for three years, then 5% annually for two more years.

- (a) Find its value after three years. [2]
- (b) Find its value after five years. [3]

**104. 🧮 GDC***[Maximum mark: 5]*

A machine loses 10% of its value each year. After three years it is worth EUR 5832.

- (a) Find its original value. [3]
- (b) Find its value after five years in total. [2]

**105. 🧮 GDC***[Maximum mark: 6]*

EUR 10000 earns a nominal 4.8% annual rate compounded monthly; inflation is 2% annually.

- (a) Find the effective annual growth factor to 8 decimal places. [2]
- (b) Find the annual real growth rate. [2]
- (c) Find its real value after five years. [2]



## 1.5 | Integer exponents and introductory logarithms

## Short response

106. NC Evaluate  $4^{-2}$ .

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107. NC Evaluate  $(-3)^3$ .

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108. NC Evaluate  $-3^4$ .

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109. NC Evaluate  $(\frac{2}{3})^{-2}$ .

---

110. NC Simplify  $x^5x^{-8}$  for  $x \neq 0$ .

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111. NC Simplify  $\frac{y^{-2}}{y^{-6}}$  for  $y \neq 0$ .

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112. NC Write  $10^{-4} = 0.0001$  in logarithmic form.

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113. NC Write  $\ln 7 = p$  in exponential form.

---

114. NC Evaluate  $\log 100000$  and  $\ln(e^{-3})$ .

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115. NC State whether  $\log(-2)$  and  $\ln 0$  have real values. Explain.

## Extended response

116. NC Simplify  $\frac{(2x^{-2}y^3)^2}{4x^{-1}y}$  for  $x, y \neq 0$ .

---

117. NC Explain the difference between  $3x^{-2}$  and  $(3x)^{-2}$ .



118. NC Complete the table of powers and logarithms.

| $x$ | $10^x$    | $\log(10^x)$ |
|-----|-----------|--------------|
| 3   | $\square$ | 3            |
| -2  | 0.01      | $\square$    |
| 0   | $\square$ | 0            |

119. NC Given  $10^t = 420$ , write  $t$  exactly and state the consecutive integers between which it lies.

120. NC Find  $y$  if  $\ln y = -4$ . Explain why  $y$  is positive.

121. NC Find the real values of  $x$  for which  $\log(3x - 6)$  is defined.

122. GDC Evaluate  $\log 72$  to 3 significant figures.

123. GDC Evaluate  $\ln 0.12$  to 3 significant figures.

124. GDC Find  $x$  when  $10^x = 570$ , giving both an exact value and a decimal approximation to 3 significant figures.

125. GDC A sensor uses  $r = \ln S$ , where  $S > 0$ . Find  $S$  when  $r = 1.8$ , to 3 significant figures.

### Exam-style

126. NC

[Maximum mark: 5]

Let  $p = \log 0.001$  and  $q = \ln(e^4)$ .

- (a) Find  $p$  and  $q$ . [2]
- (b) Find  $10^{p+q}$ . [1]
- (c) Find  $e^{pq}$  exactly. [2]

127. NC

[Maximum mark: 4]

Let  $E = \frac{(3a^{-1}b^2)^2}{9a^{-3}b}$  for  $a, b \neq 0$ .

- (a) Simplify  $E$ . [3]
- (b) Find  $E$  when  $a = -2$ ,  $b = 3$ . [1]

**128. NC***[Maximum mark: 5]*Let  $p = \ln 6$ .

- (a) Write down  $e^p$ . [1]
- (b) Find  $e^{2p}$ . [2]
- (c) Find  $e^{-p}$ . [2]

**129. NC***[Maximum mark: 5]*Let  $t = \log 250$ .

- (a) Write the statement in exponential form. [1]
- (b) Find  $10^{t+1}$ . [2]
- (c) Find  $10^{t-2}$ . [2]

**130. NC***[Maximum mark: 5]*A simplified sound model is  $L = 10 \log R$  with intensity ratio  $R > 0$ .

- (a) Find  $L$  when  $R = 10^5$ . [2]
- (b) Find  $R$  when  $L = 30$ . [2]
- (c) Interpret  $L = 0$ . [1]

**131. NC***[Maximum mark: 5]*A simplified chemistry model uses  $\text{pH} = -\log c$ , where  $c$  is a positive numerical concentration.

- (a) Find the pH when  $c = 10^{-6}$ . [1]
- (b) Find  $c$  when the pH is 4. [2]
- (c) Compare the two concentrations. [2]

**132. NC***[Maximum mark: 5]*A student claims  $x^{-2} = -x^2$  for nonzero  $x$ .

- (a) Give a numerical counterexample. [2]
- (b) Write the correct rule. [1]
- (c) Explain why  $(-x)^{-2}$  is positive. [2]

**133. GDC***[Maximum mark: 3]*The number  $x$  satisfies  $e^x = 0.04$ .

- (a) Write  $x$  exactly. [1]
- (b) Find  $x$  to 3 significant figures. [1]
- (c) Explain the sign of  $x$ . [1]

**134. NC***[Maximum mark: 5]*

For nonzero  $x, y$ , let  $E = \frac{(x^{-1}y^2)^{-2}}{x^3y^{-1}}$ .

- (a) Simplify  $E$  using positive exponents. [3]
- (b) Can  $x = 0$  or  $y = 0$  be included after simplification? Explain. [2]

**135. NC***[Maximum mark: 4]*

Consider  $\ln(8 - 2x)$ .

- (a) State its real domain. [1]
- (b) Use the definition of a logarithm to solve  $\ln(8 - 2x) = 0$ . [3]

### 1.7 | Rational exponents, logarithm laws and equations

#### Short response

**136. NC** Evaluate  $27^{2/3}$  exactly.**137. NC** Evaluate  $16^{-3/4}$  exactly.**138. NC** Evaluate  $32^{3/5}$  exactly.**139. NC** Evaluate  $(\frac{9}{16})^{-1/2}$  exactly.**140. NC** Simplify  $x^{3/2}x^{-1/2}$  for  $x > 0$ .**141. NC** Evaluate  $\log_2 32 + \log_2 \frac{1}{4}$ .**142. NC** Write  $2 \ln x - \ln y$  as one logarithm, for  $x, y > 0$ .**143. NC** Expand  $\log_3(9x^2/y)$  for  $x, y > 0$ .**144. NC** Use change of base to evaluate  $\log_8 4$  exactly.



145. NC Solve  $2^{x+1} = 16$ .

Extended response

146. NC Simplify  $\frac{(8x^6)^{2/3}}{2x}$  for  $x > 0$ .

---

147. NC Solve  $9^{x-1} = 27^{2x+1}$ .

---

148. NC Solve  $5^{2x} = 7$ , leaving an exact answer.

---

149. NC Solve  $3e^{2x} - 5 = 16$ , leaving an exact answer.

---

150. NC Solve  $\ln(x - 1) = 2$ .

---

151. NC Solve  $\log_2 x + \log_2(x - 2) = 3$ .

---

152. NC Solve  $\ln(x + 3) - \ln x = \ln 4$ .

---

153. NC Let  $p = \log 2$  and  $q = \log 3$ . Express  $\log 72$  and  $\log_3 8$  in terms of  $p, q$ .

---

154. NC Solve  $4^x - 5 \cdot 2^x + 4 = 0$ .

---

155. NC Solve  $e^{2x} - 4e^x - 5 = 0$ .

---

156. GDC Find  $\log_7 20$  to 3 significant figures.

---

157. GDC Solve  $6(1.08)^t = 15$  for real  $t$ , to 3 significant figures.

---

158. NC Find the real domain of  $\ln(x - 2) + \ln(7 - x)$ .

---



**159. NC** A student claims  $\log(x + y) = \log x + \log y$  for  $x, y > 0$ . Disprove the claim and state the correct product law.

---

**160. NC** Write  $\frac{1}{2}\ln(x + 1) - 2\ln(x - 1)$  as one logarithm, and state its original domain.

**Exam-style**

**161. NC** *[Maximum mark: 5]*  
Let  $p = \log 2$  and  $q = \log 5$ .

- (a) Express  $\log 40$  in terms of  $p, q$ . [2]
  - (b) Express  $\log_4 25$  in terms of  $p, q$ . [2]
  - (c) Show that  $p + q = 1$ . [1]
- 

**162. NC** *[Maximum mark: 5]*  
Consider  $\log_3(x + 1) + \log_3(x - 1) = 1$ .

- (a) State the domain. [1]
  - (b) Use logarithm laws to find the candidate solutions. [3]
  - (c) State the valid solution. [1]
- 

**163. NC** *[Maximum mark: 5]*  
Consider  $2^{2x} - 10 \cdot 2^x + 16 = 0$ .

- (a) Use  $t = 2^x$  to obtain a quadratic. [1]
  - (b) Solve the original equation. [4]
- 

**164. NC** *[Maximum mark: 6]*  
Let  $a > 0$  and  $\log_a 16 = 2$ .

- (a) Find  $a$ . [2]
  - (b) Evaluate  $\log_a 8$ . [2]
  - (c) Evaluate  $\log_a(1/2)$ . [2]
- 

**165.  GDC** *[Maximum mark: 6]*  
A model is  $N = 120e^{0.3t}$ , where  $t$  is measured in hours.

- (a) State the initial value. [1]
  - (b) Find when  $N = 360$ . [3]
  - (c) Find  $N$  after five hours. [2]
- 

**166. NC** *[Maximum mark: 4]*  
Solve  $\log_2(x + 6) - \log_2 x = 2$ .

- (a) State the domain. [1]
- (b) Solve the equation. [3]

**167. NC***[Maximum mark: 5]*

The number  $p$  satisfies  $\log_3 p = 4$ .

- (a) Find  $p$ . [1]
- (b) Find  $\log_9 p$ . [2]
- (c) Find  $\log_p 3$ . [2]

**168. NC***[Maximum mark: 5]*

For  $x > 0$ , let  $E = \frac{(27x^9)^{2/3}}{3x^{1/2}}$ .

- (a) Simplify  $E$ . [3]
- (b) Evaluate  $E$  when  $x = 4$ . [2]

**169. NC***[Maximum mark: 5]*

Consider  $\ln x + \ln(x + 2) = \ln 8$ .

- (a) State the domain. [1]
- (b) Solve the equation. [4]

**170.  GDC***[Maximum mark: 4]*

A quantity follows  $Q = 80(0.92)^t$  for  $t \geq 0$ .

- (a) Find an exact expression for the time at which  $Q = 40$ . [2]
- (b) Calculate this time to 3 significant figures. [1]
- (c) Find the first integer time at which  $Q < 40$ . [1]

**171. NC***[Maximum mark: 5]*

Consider  $\log_2 x = \log_4(8x)$ .

- (a) Use change of base to write the right-hand side in base 2. [2]
- (b) Hence solve the equation. [3]

**172. NC***[Maximum mark: 5]*

Let  $p = \log_2 3$  and  $q = \log_2 5$ .

- (a) Express  $\log_2 45$  in terms of  $p, q$ . [2]
- (b) Express  $\log_3(25/2)$  in terms of  $p, q$ . [3]

**173. NC***[Maximum mark: 4]*

Consider  $e^{2x} - 7e^x + 12 = 0$ .

- (a) Reduce the equation to a factorized quadratic. [2]
- (b) Give the exact solutions. [2]

**174. NC***[Maximum mark: 5]*Let  $f = \ln(x - 1) + \ln(5 - x)$  for real  $x$ .

- (a) State the domain. [1]
- (b) Find the values of  $x$  for which  $f = \ln 3$ . [4]

**175. NC***[Maximum mark: 4]*A student writes  $\log_2(2^{x+1}) = 2x$ .

- (a) Simplify the left-hand side correctly. [1]
- (b) Find when the student's equation happens to hold. [2]
- (c) Explain why it is not an identity. [1]

### 1.8 | Infinite geometric series

#### Short response

**176. NC** Find the sum to infinity of the geometric series with  $a = 6$  and  $r = \frac{1}{2}$ .**177. NC** Find the sum to infinity of the geometric series with  $a = 8$  and  $r = \frac{1}{3}$ .**178. NC** Find the sum to infinity of the geometric series with  $a = 5$  and  $r = -\frac{1}{4}$ .**179. NC** Find the sum to infinity of the geometric series with  $a = 12$  and  $r = \frac{2}{3}$ .**180. NC** Does  $3 + 6 + 12 + \dots$  have a finite sum to infinity? Explain.**181. NC** Does  $2 - 2 + 2 - 2 + \dots$  have a convergent ordinary sum? Explain.**182. NC** A convergent geometric series has first term 9 and sum 12. Find its ratio.**183. NC** A convergent geometric series has ratio  $-\frac{1}{2}$  and sum 10. Find its first term.**184. NC** Find the range of  $x$  for which  $1 + x + x^2 + \dots$  converges.



**185. NC** Express  $0.\overline{3}$  as a fraction using an infinite series.

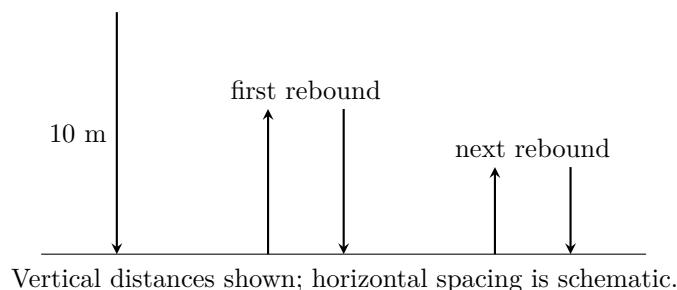
**Extended response**

**186. NC** The geometric series  $4 + 2 + 1 + \dots$  is considered.

| $n$   | 1                        | 2                        | 3                        | 4                        |
|-------|--------------------------|--------------------------|--------------------------|--------------------------|
| $S_n$ | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> |

- (a) Complete the table of partial sums.
- (b) Find the limiting sum.
- (c) Find an expression for the distance from  $S_n$  to the limiting sum.

**187. NC** A ball is dropped from 10 m. Each rebound reaches  $\frac{3}{5}$  of the preceding height.



- (a) Find the first two rebound heights.
- (b) Find the total vertical distance in the idealized infinite process.

**188. NC** Express  $0.\overline{27}$  as a fraction using an infinite geometric series.

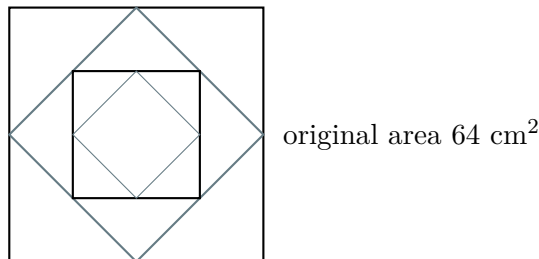
**189. NC** The first term of a geometric series is 12. Its sum to infinity is 16.

- (a) Find  $r$ .
- (b) Find  $S_3$ .

**190. NC** A geometric series has  $a = 3$  and  $r = \frac{1}{2}$ . Find the least  $n$  for which  $S_\infty - S_n < \frac{1}{10}$ .

**191. NC** Find the values of  $x$  for which  $5 + 5(2x - 1) + 5(2x - 1)^2 + \dots$  converges, and give its sum.

**192. NC** A square of area  $64 \text{ cm}^2$  has a square formed by joining its side midpoints. The construction repeats indefinitely.



- State the ratio of successive areas and the next three areas.
- Find the sum of the areas of all squares, including the original.

**193. NC** A geometric series has sum 18 and second term 4. Find all possible first terms and ratios.

**194. NC** Find the sum  $\sum_{k=3}^{\infty} 2^{-k}$ .

**195. NC** A geometric series has  $a = 7$  and  $r = -\frac{1}{3}$ . Find  $S_{\infty}$  and  $|S_{\infty} - S_4|$ .

### Exam-style

**196. NC** *[Maximum mark: 5]*  
A geometric series has first term 20 and ratio  $\frac{3}{5}$ .

- Find its sum to infinity. [1]
- Find  $S_4$ . [3]
- Find the difference between  $S_{\infty}$  and  $S_4$ . [1]

**197. NC** *[Maximum mark: 6]*  
The infinite series is  $6 + 6(x + 1) + 6(x + 1)^2 + \dots$ .

- Find the interval of convergence. [2]
- Find its sum in terms of  $x$ . [2]
- Find  $x$  if the sum is 12. [2]

**198. NC** *[Maximum mark: 5]*  
A ball is dropped from 12 m and each rebound reaches half the previous height.

- Find the height of the  $n$ th rebound. [2]
- Find the total distance in the infinite model. [3]

**199. NC***[Maximum mark: 5]*

A convergent geometric series has  $S_{\infty} = 24$  and  $S_2 = 18$ .

- (a) Find all possible common ratios. [3]
- (b) Find the corresponding first terms. [2]

**200.  GDC***[Maximum mark: 5]*

A geometric series has  $a = 5$  and  $r = 0.8$ .

- (a) Find  $S_{\infty}$ . [1]
- (b) Find the least  $n$  for which  $|S_{\infty} - S_n| < 0.01$ . [4]

**201. NC***[Maximum mark: 4]*

Consider  $0.\overline{18}$ .

- (a) Write it as an infinite geometric series. [2]
- (b) Hence express it as a fraction. [2]

**202. NC***[Maximum mark: 5]*

Two infinite geometric series have first term 4 and ratios  $r$  and  $-r$ , where  $0 < r < 1$ . Their sums are 12 and 6, respectively.

- (a) Find the value of  $r$  implied by the first sum. [2]
- (b) Determine whether both given sums can be correct. [3]

**203. NC***[Maximum mark: 5]*

An infinite geometric series has first term  $a > 0$  and ratio  $\frac{1}{4}$ . Its sum is 20.

- (a) Find  $a$ . [1]
- (b) Find the sum of the first three terms. [2]
- (c) Find the fraction of the infinite sum remaining after three terms. [2]

**204. NC***[Maximum mark: 5]*

An infinite geometric series has first term 2 and ratio  $x^2$ .

- (a) Find the values of  $x$  for which the series converges. [2]
- (b) Find  $x$  when the sum is  $8/3$ . [3]

**205. NC***[Maximum mark: 6]*

A sequence of partial sums is  $S_n = 10[1 - (-\frac{1}{2})^n]$ .

- (a) Find the first term and common ratio of the underlying series. [3]
- (b) Find its sum to infinity. [1]
- (c) Explain why the partial sums alternate above and below the limit. [2]



## 1.9 | Binomial theorem: nonnegative integer powers

## Short response

206. NC Evaluate  $\binom{5}{2}$  without a calculator.

---

207. NC Evaluate  $\binom{6}{3}$  without a calculator.

---

208. NC Evaluate  $\binom{8}{2}$  without a calculator.

---

209. NC Expand  $(1 + x)^4$ .

---

210. NC Expand  $(1 - x)^5$ .

---

211. NC Expand  $(2 + x)^3$ .

---

212. NC Find the coefficient of  $x^2$  in  $(1 + 3x)^4$ .

---

213. NC Find the coefficient of  $x^3$  in  $(2 - x)^5$ .

---

214. NC Write the general term in the expansion of  $(a + b)^n$ , where  $n$  is a positive integer.

---

215. NC Complete the next row of Pascal's triangle.

| $n$ | Coefficients |   |           |   |   |
|-----|--------------|---|-----------|---|---|
| 2   |              | 1 | 2         | 1 |   |
| 3   |              | 1 | 3         | 3 | 1 |
| 4   | 1            | 4 | 6         | 4 | 1 |
| 5   |              |   | $\square$ |   |   |

## Extended response

216. NC Find the first four terms in ascending powers of  $x$  of  $(1 - 2x)^6$ .

---

217. NC Find the coefficient of  $x^4$  in  $(2 + 3x)^6$ .

---



---

**218. NC** Find the term independent of  $x$  in  $(x + \frac{2}{x})^4$ .

---

**219. NC** Find the coefficient of  $x^2$  in  $(1 + x)^3(1 - 2x)^2$ .

---

**220. NC** The coefficient of  $x$  in  $(1 + kx)^5$  is 15. Find  $k$  and the coefficient of  $x^2$ .

---

**221. NC** Find the two middle terms in  $(1 + x)^5$ .

---

**222. NC** Find the middle term in  $(2x - 1)^4$ .

---

**223. NC** Use the binomial theorem to calculate  $101^3$  exactly.

---

**224. NC** Find all integers  $r$  with  $0 \leq r \leq 7$  for which  $\binom{7}{r} = 21$ .

---

**225. GDC** Find  $\binom{30}{12}$  using technology.

### Exam-style

**226. NC**

[Maximum mark: 5]

Consider  $(1 + 2x)^5$ .

(a) Find the first four terms in ascending powers of  $x$ . [3]

(b) Find the coefficient of  $x^4$ . [2]

---

**227. NC**

[Maximum mark: 5]

The coefficient of  $x^2$  in  $(1 + kx)^4$  is 54, where  $k > 0$ .

(a) Find  $k$ . [3]

(b) Find the coefficient of  $x^3$ . [2]

---

**228. NC**

[Maximum mark: 6]

Consider  $(2x - \frac{1}{x})^6$ , with  $x \neq 0$ .

(a) Write the general term. [2]

(b) Find the constant term. [2]

(c) Find the coefficient of  $x^2$ . [2]

---

**229. NC***[Maximum mark: 4]*Let  $P(x) = (1 + x)^4(1 - 2x)$ .

- (a) Find the coefficient of  $x^2$ . [2]  
(b) Find the coefficient of  $x^3$ . [2]

**230. NC***[Maximum mark: 5]*For a positive integer  $n$ , the coefficient of  $x$  in  $(1 + x)^n$  is 8.

- (a) Find  $n$ . [1]  
(b) Find the coefficient of  $x^2$ . [2]  
(c) Find the sum of all coefficients. [2]

**231. NC***[Maximum mark: 5]*The coefficient of  $x^2$  in  $(1 + x)^n$  is 15, where  $n$  is a positive integer.

- (a) Find  $n$ . [3]  
(b) Find the coefficient of  $x^3$ . [2]

**232. NC***[Maximum mark: 6]*Consider  $(3 - x)^4$ .

- (a) Expand fully. [4]  
(b) Find the sum of its coefficients without adding them individually. [2]

**233. NC***[Maximum mark: 5]*Let  $P(x) = (1 + x)^6$ .

- (a) Find  $P(1)$  and  $P(-1)$ . [2]  
(b) Hence find the sums of the coefficients of the even and odd powers separately. [3]

**234. NC***[Maximum mark: 6]*Consider  $(x^2 + \frac{2}{x})^6$  for  $x \neq 0$ .

- (a) Write the general term. [2]  
(b) Find the constant term. [2]  
(c) Find the coefficient of  $x^3$ . [2]

**235. NC***[Maximum mark: 4]*Consider  $(2 + x)^7$ .

- (a) State which two terms are the middle terms. [1]  
(b) Find both terms. [3]

**236. NC***[Maximum mark: 5]*

The coefficient of  $x^3$  in  $(1 + kx)^5$  is  $-80$ .

- (a) Find the real value of  $k$ . [3]
- (b) Find the coefficient of  $x^2$ . [2]

**237. NC***[Maximum mark: 6]*

Let  $P(x) = (1 - 3x)^4$ .

- (a) Find terms through  $x^2$ . [2]
- (b) Use them to estimate  $0.97^4$ . [2]
- (c) Use the remaining terms to determine whether your estimate is above or below the exact value. [2]

**238. GDC***[Maximum mark: 3]*

Find the coefficient of  $x^8$  in  $(1 + 2x)^{15}$ .

- (a) Write an exact combinatorial expression for the coefficient. [2]
- (b) Evaluate it using technology. [1]

**239. NC***[Maximum mark: 6]*

Let  $P(x) = (1 + ax)^3(1 + bx)$ , with real constants  $a, b$ . The coefficients of  $x$  and  $x^2$  are 5 and 9.

- (a) Write two equations for  $a, b$ . [2]
- (b) Find all pairs  $(a, b)$ . [4]

**240. NC***[Maximum mark: 7]*

Consider  $(1 + x)^5$ .

- (a) Write its expansion. [3]
- (b) Find the ordinary and alternating sums of its coefficients. [2]
- (c) Interpret these two sums in terms of even and odd powers. [2]

**AHL 1.10 | HL exclusive: binomial series with negative and rational exponents**

**HL exclusive.** Generalized binomial expansions use  $(1 + u)^p$  and the standard guarantee  $|u| < 1$ . State this open interval when asked; endpoint analysis is not required. A finite truncation is an approximation: convergence alone does not guarantee a particular number of correct digits.

**Short response**

**241. NC** Expand  $(1 + x)^{-1}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**242. NC** Expand  $(1 - x)^{-2}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**243. NC** Expand  $(1 + 2x)^{-3}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**244. NC** Expand  $(1 + x)^{\frac{1}{2}}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**245. NC** Expand  $(1 - 2x)^{-\frac{1}{2}}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**246. NC** Expand  $(1 - x)^{\frac{3}{2}}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**247. NC** Expand  $(1 + 3x)^{\frac{1}{3}}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**248. NC** Expand  $(1 - x)^{-\frac{1}{3}}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**249. NC** Expand  $(1 + x)^{\frac{2}{3}}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**250. NC** Expand  $(1 + \frac{1}{2}x)^{-2}$  through the term in  $x^3$ . State the open interval guaranteed by the binomial condition.

**251. NC** Expand  $(4 + x)^{1/2}$  through  $x^2$ , first factoring out the constant. State the guaranteed open interval.



**252. NC** Expand  $(9 - 3x)^{-1/2}$  through  $x^2$ , first factoring out the constant. State the guaranteed open interval.

---

**253. NC** Expand  $(8 + x)^{1/3}$  through  $x^2$ , first factoring out the constant. State the guaranteed open interval.

---

**254. NC** Expand  $(16 - 8x)^{3/4}$  through  $x^2$ , first factoring out the constant. State the guaranteed open interval.

---

**255. NC** Find the coefficient of  $x^4$  in the binomial series for  $(1 + x)^{-2}$ .

---

**256. NC** Find the coefficient of  $x^4$  in the binomial series for  $(1 + 2x)^{\frac{1}{2}}$ .

---

**257. NC** Find the coefficient of  $x^3$  in the binomial series for  $(1 - x)^{-\frac{3}{2}}$ .

---

**258. NC** Find the coefficient of  $x^4$  in the binomial series for  $(1 + 3x)^{\frac{1}{3}}$ .

---

**259. NC** For the binomial series of  $(1 - 4x)^{-1/2}$ , find the guaranteed open interval and decide whether  $x = \frac{1}{5}$  lies inside it.

---

**260. NC** For the expansion of  $(1 + \frac{x-1}{2})^{1/3}$  in powers of  $(x - 1)$ , state the guaranteed open interval of  $x$ .

#### Extended response

**261. NC** Use a quadratic binomial approximation to estimate  $\sqrt{1.04}$ . Give a rational approximation; no decimal evaluation is required.

---

**262. NC** Use a quadratic binomial approximation to estimate  $\frac{1}{\sqrt{0.96}}$ . Give a rational approximation; no decimal evaluation is required.

---

**263. NC** Use a quadratic binomial approximation to estimate  $\sqrt[3]{1.06}$ . Give a rational approximation; no decimal evaluation is required.

---

**264. NC** Use a quadratic binomial approximation to estimate  $\frac{1}{(1.1)^2}$ . Give a rational approximation; no decimal evaluation is required.

---

**265. NC** Use a quadratic binomial approximation to estimate  $\sqrt{3.96}$ . Give a rational approximation; no decimal evaluation is required.

---



**266. GDC** Use the cubic binomial approximation for  $(1+x)^{\frac{1}{2}}$  at  $x = 0.2$ .

- (a) Find the cubic approximation.
- (b) Calculate the approximation and the true value to 8 decimal places.
- (c) Find the absolute error and state whether it is an overestimate or underestimate.

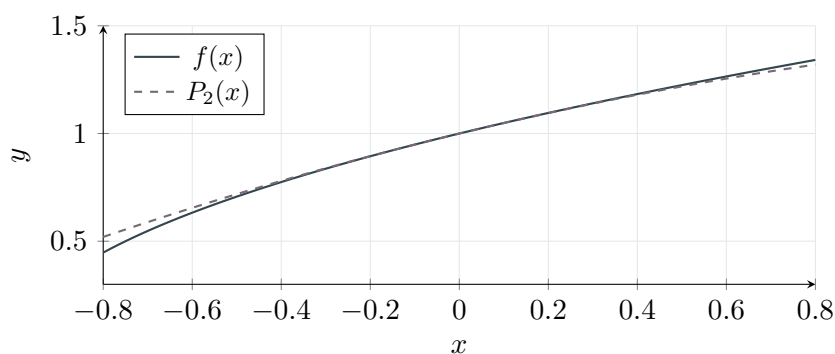
**267. GDC** Use the cubic binomial approximation for  $(1+x)^{-\frac{1}{2}}$  at  $x = 0.1$ .

- (a) Find the cubic approximation.
- (b) Calculate the approximation and the true value to 8 decimal places.
- (c) Find the absolute error and state whether it is an overestimate or underestimate.

**268. GDC** Use the cubic binomial approximation for  $(1+x)^{\frac{1}{3}}$  at  $x = -0.15$ .

- (a) Find the cubic approximation.
- (b) Calculate the approximation and the true value to 8 decimal places.
- (c) Find the absolute error and state whether it is an overestimate or underestimate.

**269. NC** The graph compares  $f(x) = \sqrt{1+x}$  with  $P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$ .



- (a) Explain how the polynomial is obtained.
- (b) Describe where the approximation appears most accurate.
- (c) Does being inside  $|x| < 1$  alone guarantee that three terms give high accuracy?

**270. NC** Expand  $(1+2x)(1-x)^{-2}$  through  $x^3$  and state a guaranteed open interval.

**271. NC** Expand  $(1+x)^{1/2}(1-x)^{-1}$  through  $x^3$  and state a guaranteed open interval.

**272. NC** Expand  $(1+x)^{-1/2}(1-x)^{-1/2}$  through  $x^3$  and state a guaranteed open interval.



**273. NC** Expand  $(1 - 3x)(1 + 2x)^{-1/2}$  through  $x^3$  and state a guaranteed open interval.

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**274. NC** Expand  $(1 + x + x^2)(1 - x)^{-1}$  through  $x^3$  and state a guaranteed open interval.

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**275. NC** The coefficient of  $x$  in  $(1 + 2x)^p$  is  $-3$ . Find  $p$  and the coefficient of  $x^2$ .

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**276. NC** The expansion of  $(1 + ax)^{-2}$  starts  $1 + 6x + \dots$ . Find  $a$  and the next two coefficients.

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**277. NC** The coefficient of  $x^2$  in  $(1 + x)^p$  is  $\frac{3}{8}$ . Find all possible  $p$  and state the corresponding coefficient of  $x$ .

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**278. NC** A student uses  $\sqrt{4 + x} = 2 + x - \frac{x^2}{4} + \dots$ . Correct the expansion through  $x^2$  and state the source of the error.

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**279. NC** Consider the expansion of  $(1 - 3x)^{-2}$ . Decide whether the standard binomial convergence condition guarantees convergence at  $x = 0.2$ ,  $x = 0.4$  and  $x = \frac{1}{3}$ .

---

**280. NC** Use the binomial theorem to write the coefficient of  $x^r$  in  $(1 - x)^{-2}$  for an integer  $r \geq 0$ . Hence identify its first five coefficients.

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### Exam-style

**281. NC**

[Maximum mark: 7]

Let  $f(x) = (1 - 2x)^{-1}$ .

- (a) Find the first four terms of its binomial expansion. [4]
  - (b) State the guaranteed open interval of convergence. [1]
  - (c) Use the four terms to estimate  $f(\frac{1}{20})$ . Leave the answer as a sum of exact fractions; you need not combine them. [2]
- 

**282. NC**

[Maximum mark: 7]

Let  $f(x) = (1 + 2x)^{1/2}$ .

- (a) Find the first four terms of its binomial expansion. [4]
- (b) State the guaranteed open interval of convergence. [1]
- (c) Use the four terms to estimate  $f(\frac{1}{50})$ . Leave the answer as a sum of exact fractions; you need not combine them. [2]

**283. NC***[Maximum mark: 7]*Let  $f(x) = (4 - x)^{-1/2}$ .

- (a) Find the first four terms of its binomial expansion. [4]
- (b) State the guaranteed open interval of convergence. [1]
- (c) Use the four terms to estimate  $f(\frac{1}{5})$ . Leave the answer as a sum of exact fractions; you need not combine them. [2]

**284. NC***[Maximum mark: 7]*Let  $f(x) = (8 + 2x)^{1/3}$ .

- (a) Find the first four terms of its binomial expansion. [4]
- (b) State the guaranteed open interval of convergence. [1]
- (c) Use the four terms to estimate  $f(\frac{1}{10})$ . Leave the answer as a sum of exact fractions; you need not combine them. [2]

**285. NC***[Maximum mark: 7]*Let  $f(x) = (1 - x)^{3/2}$ .

- (a) Find the first four terms of its binomial expansion. [4]
- (b) State the guaranteed open interval of convergence. [1]
- (c) Use the four terms to estimate  $f(\frac{1}{10})$ . Leave the answer as a sum of exact fractions; you need not combine them. [2]

**286. NC***[Maximum mark: 8]*The binomial expansion of  $(1 + ax)^p$  begins  $1 - 2x + 6x^2 + \dots$ , where  $a, p$  are nonzero real numbers.

- (a) Write two equations for  $a, p$ . [2]
- (b) Find  $a, p$ . [3]
- (c) Find the coefficient of  $x^3$  and state the guaranteed interval. [3]

**287. NC***[Maximum mark: 6]*Consider  $f(x) = (1 + x)^{-1/2}(1 - x)^{-1/2}$ .

- (a) Write  $f$  as one negative power. [2]
- (b) Find the expansion through  $x^4$ . [3]
- (c) Explain why the coefficient of  $x^3$  is zero. [1]

**288. NC***[Maximum mark: 7]*Let  $f(x) = (1 + 3x)^{-2}$ .

- (a) Expand through  $x^3$ . [3]
- (b) Find the coefficient of  $x^r$  for an integer  $r \geq 0$ . [3]
- (c) Hence find the coefficient of  $x^4$ . [1]

**289. NC***[Maximum mark: 7]*

Let  $f(x) = (1 + ax)(1 - x)^{-1/2}$ . The coefficient of  $x$  is zero.

- (a) Expand the negative power through  $x^3$ . [3]
- (b) Find  $a$ . [2]
- (c) Find  $f(x)$  through  $x^3$ . [2]

**290. NC***[Maximum mark: 8]*

Let  $f(x) = (1 - 2x)^{1/2}(1 + x)^{-1}$ .

- (a) Expand each factor through  $x^3$ . [4]
- (b) Hence expand  $f$  through  $x^3$ . [3]
- (c) State a common guaranteed open interval. [1]

**291. NC***[Maximum mark: 6]*

A sphere's volume is increased by a fraction  $x$  of its original volume. Its original radius is  $r_0$ .

- (a) Show that the new radius is  $r_0(1 + x)^{1/3}$ . [2]
- (b) Find a quadratic approximation to  $r/r_0$ . [2]
- (c) Estimate the percentage radius increase for a 6% volume increase. [2]

**292. NC***[Maximum mark: 5]*

A square's area increases by a fraction  $x$  of its original area. Its side-length ratio is  $R$ .

- (a) Express  $R$  in terms of  $x$ . [1]
- (b) Find a quadratic approximation. [2]
- (c) Estimate the percentage side-length increase for a 4% area increase. [2]

**293.  GDC***[Maximum mark: 7]*

For  $f(x) = (1 + x)^{-2}$ , consider the quadratic approximation  $P_2(x) = 1 - 2x + 3x^2$ .

- (a) Calculate  $P_2(0.1)$  and  $f(0.1)$ . [2]
- (b) Find the absolute and percentage errors. [3]
- (c) Compare with the approximation at  $x = 0.5$  and comment. [2]

**294. NC***[Maximum mark: 6]*

The expansion of  $(1 + x)^p$  has equal coefficients of  $x^2$  and  $x^3$ . Assume  $p \notin \{0, 1\}$ .

- (a) Write an equation for  $p$ . [2]
- (b) Find  $p$ . [2]
- (c) Explain what this means for a search restricted to negative or noninteger exponents. [2]

**295. NC***[Maximum mark: 7]*Let  $f(x) = (1 - 4x^2)^{-1/2}$ .

- (a) Find the first three nonzero terms. [3]
- (b) State the guaranteed interval. [2]
- (c) Use these terms to estimate  $f(1/10)$  as a fraction. [2]

**296. NC***[Maximum mark: 8]*Let  $f(x) = \frac{1+x}{(1-x)^2}$ .

- (a) Expand the negative power through  $x^3$ . [3]
- (b) Find the first four terms of  $f$  and a general coefficient formula. [3]
- (c) Find the exact sum  $1 + 3/4 + 5/16 + 7/64 + \dots$ . [2]

**297. NC***[Maximum mark: 7]*The coefficient of  $x^2$  in  $(1 + kx)^{-1/2}$  is 6, with  $k < 0$ .

- (a) Find  $k$ . [3]
- (b) Expand through  $x^3$ . [3]
- (c) State the guaranteed interval. [1]

**298. NC***[Maximum mark: 8]*Let  $f(x) = (1 + x)^{1/2} + (1 - x)^{1/2}$ .

- (a) Find each expansion through  $x^4$ . [4]
- (b) Hence find  $f(x)$  through  $x^4$ . [2]
- (c) Explain the missing odd powers and state the interval. [2]

**299. NC***[Maximum mark: 6]*Let  $g(x) = (1 + x)^{1/2} - (1 - x)^{1/2}$ .

- (a) Find the first two nonzero terms. [3]
- (b) Use them to estimate  $\sqrt{1.2} - \sqrt{0.8}$  as a fraction. [2]
- (c) Explain why there is no  $x^2$  term. [1]

**300. NC***[Maximum mark: 9]*Let  $f(x) = (4 + 4x)^{-1/2}(1 - 2x)^{-1}$ .

- (a) Rewrite the expression using factors with constant term 1. [1]
- (b) Expand through  $x^3$ . [5]
- (c) State a guaranteed interval. [1]
- (d) Estimate  $f(1/10)$  using your cubic polynomial. [2]



## Answer key and solution outlines

Question numbers and subparts match the student sections. Exact expressions remain exact; decimal answers are rounded only at the end. For GDC thresholds, the least whole period is obtained by checking the first integer beyond the strict bound.

## 1.2 | Arithmetic sequences and series

## Short response

1.  $u_n = u_1 + (n - 1)d$ , so  $u_{12} = 37$ .
2.  $u_n = u_1 + (n - 1)d$ , so  $u_{10} = 7$ .
3.  $u_n = u_1 + (n - 1)d$ , so  $u_9 = 33$ .
4.  $u_n = u_1 + (n - 1)d$ , so  $u_8 = 11$ .
5.  $d = -4$ ;  $u_n = 9 - 4(n - 1) = 13 - 4n$ .
6.  $5d = 20$ , so  $d = 4$  and  $u_1 = 17 - 3(4) = 5$ .
7.  $S_{20} = \frac{20}{2}[2(3) + 19(4)] = 820$ .
8. First term 3, last term 21; sum  $= \frac{10}{2}(3 + 21) = 120$ .
9.  $\sum_{k=1}^{10} (4k + 2)$ .
10.  $u_n = 35 - 4n < 0 \Rightarrow n > 8.75$ ;  $u_9 = -1$ .

## Extended response

11.

- (a)  $u_1 = 5$  and  $d = 3$ .
- (b)  $u_n = 3n + 2$ .
- (c)  $u_{20} = 62$ .

12.  $u_1 = 7$ ,  $d = 5$ . Missing values: 12, 22, 27;  $u_{15} = 77$ .

13.

- (a)  $u_{12} = 18 + 11(3) = 51$  seats.
- (b)  $S_{12} = 6(18 + 51) = 414$  seats.

14.

- (a)  $u_{15} = 8 + 14(2) = 36$  euros.
- (b)  $S_n = n(n + 7)$ ;  $S_{11} = 198$ ,  $S_{12} = 228$ , so 12 weeks.

15. Equal differences:  $2x - 3 = x + 4$ , so  $x = 7$ ; the terms are 8, 19, 30.16. There are five equal gaps:  $d = (28 - 3)/5 = 5$ . Insert 8, 13, 18, 23.



## 1.2 | Arithmetic sequences and series (continued)

17.

- (a)  $u_1 = S_1 = 5$ .  
(b)  $u_n = S_n - S_{n-1} = 4n + 1$  for  $n \geq 2$ ; also valid for  $n = 1$ .  
(c)  $d = 4$ , so the sequence is arithmetic.

18. There are  $90 - 40 + 1 = 51$  terms; sum  $= \frac{51}{2}(40 + 90) = 3315$ .

19. GDC

- (a) Average change  $= (25.9 - 14.1)/4 = 2.95$ ;  $u_n = 14.1 + 2.95(n - 1)$ .  
(b)  $u_8 = 34.75 \approx 34.8$  (3 s.f.).  
(c) The observed differences are not exactly equal, so predictions are approximate.

20.  $a + 4d = 14$  and  $2a + 9d = 31$ . Subtracting twice the first equation gives  $d = 3$ ;  $a = 2$ .**Exam-style**

21.

- (a)  $5d = 20 \Rightarrow d = 4$ ;  $a = 3$ .  
(b)  $S_{20} = 10[6 + 19(4)] = 820$ .

22.

- (a)  $u_n = 2n + 3$ .  
(b)  $S_n = n(n + 4)$ .  
(c)  $n(n + 4) = 221 \Rightarrow (n - 13)(n + 17) = 0$ ; 13 rows.

23.

- (a)  $42 - 3(n - 1) > 0 \Rightarrow n < 15$ : 14 positive terms.  
(b)  $S_{14} = 7(42 + 3) = 315$ .

24.

- (a)  $2a + 9d = 34$ ,  $2a + 19d = 64$ , hence  $d = 3$ ,  $a = \frac{7}{2}$ .  
(b)  $u_{30} = \frac{7}{2} + 87 = \frac{181}{2}$ .

25.

- (a)  $u_5 - u_2 = 3d = p$ ;  $u_9 = 2p + 4p/3 = 28$ , so  $p = \frac{42}{5}$ .  
(b)  $d = \frac{14}{5}$ ;  $a = p - d = \frac{28}{5}$ .

26.

- (a)  $u_1 = 5$ ,  $d = -2$ .  
(b)  $\sum_{n=4}^{12} u_n = \frac{9}{2}(-1 - 17) = -81$ .

27.

- (a) Year 8 salary  $= 30000 + 7(1500) = 40500$ .  
(b) Total  $= \frac{8}{2}(30000 + 40500) = 282000$  euros.



## 1.2 | Arithmetic sequences and series (continued)

**28.**

- (a)  $S_n = \frac{n}{4}(n + 7)$ .  
(b)  $S_{16} = 92$ ,  $S_{17} = 102$ ; day 17.

**29.**

- (a)  $u_1 = 1$ ,  $u_n = 6n - 5$ , so  $d = 6$ .  
(b)  $6n - 5 = 73 \Rightarrow n = 13$ .

**30.**

- (a)  $S_n = \frac{n}{2}(5n - 29)$ .  
(b) For  $n > 0$ ,  $S_n > 0$  iff  $n > 29/5$ ; least  $n = 6$ .

**31.**

- (a)  $S_n = \frac{n}{2}[2 + 2(n - 1)] = n^2$ .  
(b)  $1 + 3 + \cdots + 39 = 20^2 = 400$ .  
(c)  $n^2 = 625 \Rightarrow n = 25$ .

**32.**

- (a)  $u_{p+6} = 17 + 6(4) = 41$ .  
(b) Seven terms; sum  $= \frac{7}{2}(17 + 41) = 203$ .

**33.**

- (a)  $u_{20} = 194$ .  
(b)  $S_{20} = 10(80 + 194) = 2740$ .  
(c) Demand or venue capacity may prevent the same increase from continuing.

**34.**

- (a)  $4n + 1 = 7n - 14 \Rightarrow n = 5$ ; common term 21.  
(b)  $\sum_{n=1}^5 a_n = \frac{5}{2}(5 + 21) = 65$ .

**35.**

- (a)  $180 = \frac{15}{2}(a + 26) \Rightarrow a = -2$ .  
(b)  $d = (26 + 2)/14 = 2$ .  
(c)  $u_8 = -2 + 7(2) = 12$ , equal to  $180/15$ .



## 1.3 | Geometric sequences and finite series

## Short response

36.  $u_n = ar^{n-1}$  gives  $u_7 = 192$ .
37.  $u_n = ar^{n-1}$  gives  $u_6 = 2$ .
38.  $u_n = ar^{n-1}$  gives  $u_5 = 162$ .
39.  $u_n = ar^{n-1}$  gives  $u_5 = 1$ .
40.  $r = -\frac{1}{2}$ ;  $u_n = 12(-\frac{1}{2})^{n-1}$ .
41.  $S_6 = 2(3^6 - 1)/(3 - 1) = 728$ .
42.  $S_5 = 48(1 - (1/2)^5)/(1 - 1/2) = 93$ .
43. First term 1, six terms; sum  $= 2^6 - 1 = 63$ .
44.  $\sum_{k=0}^5 5 \cdot 3^k$ .
45. Pairs cancel and one 4 remains:  $S_{11} = 4$ .

## Extended response

46.

- (a)  $r^3 = 8$ , so  $r = 2$ ;  $a = 6$ .
- (b)  $S_5 = 6(2^5 - 1) = 186$ .

47.

- (a)  $r^2 = 9$ , so  $r = 3$  or  $-3$ .
- (b)  $u_4 = 135$  or  $-135$ , respectively.

48.

- (a)  $a = 2$ ,  $r = 2$ .
- (b)  $u_7 = 128$ .

49.  $r = 3$ ; missing entries 9, 81;  $u_n = 3^n$ .

50. GDC

- (a)  $P_0 = 200$ ; annual increase 15%.
- (b)  $P_6 = 200(1.15)^6 = 463$ , to the nearest whole individual.

51. GDC

- (a)  $M_n = 50(0.8)^n$ .
- (b)  $M_5 = 16.4$  g (3 s.f.).

52.  $4r^3 = 108 \Rightarrow r^3 = 27 \Rightarrow r = 3$ . Insert 12, 36.53. All terms are 7;  $u_{20} = 7$ ,  $S_{20} = 140$ . The quotient formula would divide by  $1 - r = 0$ .54. First selected term 24, five terms; sum  $= 24(2^5 - 1) = 744$ .55.  $u_n = S_n - S_{n-1} = 5 \cdot 2^{n-1}$ , also valid at  $n = 1$ ;  $a = 5$ ,  $r = 2$ .



## 1.3 | Geometric sequences and finite series (continued)

**Exam-style****56.**

- (a)  $u_n = 6 \cdot 3^{n-1}$ .  
(b)  $6 \cdot 3^{n-1} = 1458 \Rightarrow 3^{n-1} = 243 = 3^5$ , so  $n = 6$ .  
(c)  $S_6 = 6(3^6 - 1)/2 = 2184$ .

**57.**

- (a)  $r^3 = 27 \Rightarrow r = 3$ ;  $a = 6$ .  
(b)  $S_4 = 6(81 - 1)/2 = 240$ .

**58.**

- (a)  $(x + 6)^2 = x(x + 18) \Rightarrow x = 6$ .  
(b)  $r = 2$ ;  $S_6 = 6(64 - 1) = 378$ .

**59.**

- (a)  $96(1/2)^{n-1} = 3 \Rightarrow 2^{n-1} = 32$ , so  $n = 6$ .  
(b)  $S_6 = 96(1 - 1/64)/(1/2) = 189$ .

**60.**

- (a)  $r = \pm 3$ .  
(b) For  $r = 3$ ,  $S_4 = 80$ ; for  $r = -3$ ,  $S_4 = -40$ .

**61.**

- (a)  $u_6 = 3 \cdot 3^5 = 729$ .  
(b)  $S_6 = 3(3^6 - 1)/2 = 1092$ .  
(c) The model assumes new viewers do not overlap and sharing continues unchanged.

**62.**

- (a)  $u_4 = -1$ ,  $u_5 = \frac{1}{2}$ .  
(b)  $S_6 = 8(1 - 1/64)/(1 + 1/2) = \frac{21}{4}$ .

**63.  GDC**

- (a)  $N_t = 500(1.18)^t$ .  
(b)  $t > \ln(4)/\ln(1.18) = 8.3757$ ; first whole hour  $t = 9$ .

**64.**

- (a)  $a(1 + 2 + 4) = 21 \Rightarrow a = 3$ .  
(b)  $S_6 = 3(64 - 1) = 189$ .

**65.**

- (a)  $u_3 = 36$ .  
(b) Six terms,  $u_3$  to  $u_8$ ; sum  $= 36(3^6 - 1)/2 = 13104$ .



## 1.3 | Geometric sequences and finite series (continued)

**66.**

(a)  $u_4 = 160(3/4)^3 = \frac{135}{2}$  units in the model.

(b)  $S_4 = 160 + 120 + 90 + 67.5 = \frac{875}{2}$  units.

**67.**

(a) Arithmetic: 11; geometric: 16.

(b) Arithmetic sum = 26; geometric sum = 30.

(c) Equal first and third terms do not determine the sequence type.

**68.**

(a)  $r^4 = 16 \Rightarrow r = 2$  or  $-2$ .

(b)  $a = 20/r^2 = 5$  for both.

(c)  $u_6 = 160$  or  $-160$ .

**69.**

(a)  $S_n = 5(2^n - 1)$ .

(b)  $S_n > 300 \Rightarrow 2^n > 61$ ;  $2^5 = 32$ ,  $2^6 = 64$ , so  $n = 6$ .

**70.**

(a)  $u_1 = S_1 = 8$ .

(b)  $u_n = 12(3^{-(n-1)} - 3^{-n}) = 8 \cdot 3^{-(n-1)}$ .

(c)  $r = \frac{1}{3}$ .

## 1.4 | Financial mathematics

**Short response**

**71.** Interest =  $2000(0.05)(3) = 300$  euros; final amount EUR 2300.

**72.**  $V = 1000(1.1)^2 = 1210$  euros.

**73.**  $V = 800(3/4)^2 = 450$  euros.

**74.** Monthly rate =  $6/12 = 0.5\%$ ;  $12(5) = 60$  periods.

**75.**  $V = 5000(1 + 0.04/4)^{4t} = 5000(1.01)^{4t}$ .

**76.**  $(1 + i)^2 = 484/400 = 121/100$ , so  $1 + i = 11/10$ ; annual rate 10%.

**77.**  $240(5/4)^2 = 375$  euros.

**78.** Real value =  $1210/(1.1)^2 = 1000$  euros.

**79.**  **GDC**  $V = 3500(1 + 0.042/12)^{72} = \text{EUR } 4\,501.11$ .

**80.**  **GDC**  $V = 7200(1 + 0.036/4)^{20} = \text{EUR } 8\,613.03$ .



## 1.4 | Financial mathematics (continued)

## Extended response

## 81. GDC

- (a) Simple:  $2000(1 + 0.05 \cdot 3) = 2300$  euros.  
(b) Compound:  $2000(1.05)^3 = 2315.25$  euros.  
(c) Compound gives EUR 15.25 more.

## 82. GDC

- (a)  $V_5 = 24000(.82)^5 = \text{EUR } 8\,897.76$ .  
(b)  $V_n = 24000(.82)^n$ .

83. GDC  $P = 12000/(1.045)^8 = \text{EUR } 8\,438.22$ .84. GDC  $i = (6800/5000)^{1/6} - 1$ ; rate = 5.26%.85. GDC  $n > \ln(1.5)/\ln(1.038) = 10.87161$ ; first whole year 11.86. GDC  $100[(1 + 0.06/12)^{12} - 1] = 6.17\%$ .

## 87. GDC

- (a) Real annual growth =  $100(1.05/1.02 - 1) = 2.94\%$ .  
(b) Real value =  $8000(1.05/1.02)^4 = \text{EUR } 8\,983.52$ .

## 88. GDC

- (a) A: 4.8000%; B: 4.8026%.  
(b) Account B has the higher effective annual rate.

89. GDC  $(1 - d)^4 = 6144/15000 = 0.4096$ ;  $1 - d = 0.8$ , so  $d = 20\%$ .90. GDC  $N = 8$ ,  $I\% = 4$ ,  $PV = -3000$ ,  $Pmt = 0$ ,  $PpY = CpY = 2$ ; solve  $FV = 3000(1.02)^8 = 3514.98$  euros. Opposite signs for PV and FV reflect cash-flow direction.

## Exam-style

## 91. GDC

- (a)  $V_n = 6000(1.003)^{12n}$ .  
(b)  $V_7 = \text{EUR } 7\,716.66$ .  
(c) Interest = EUR 1 716.66.

## 92. GDC

- (a) After 6 years:  $28000(.84)^6 = \text{EUR } 9\,836.34$ .  
(b)  $n > \ln(10000/28000)/\ln(.84) = 5.9054$ ; first year 6.

## 93. GDC

- (a) Rate =  $100[(11/9)^{1/5} - 1] = 4.10\%$ .  
(b) At the same rate, after 10 years:  $9000(11/9)^2 = \text{EUR } 13\,444.44$ .

**1.4 | Financial mathematics (continued)****94. GDC**

- (a) Nominal value =  $10000(1.04)^8 = \text{EUR } 13\,685.69$ .  
(b) Real value =  $10000(1.04/1.025)^8 = \text{EUR } 11\,232.48$ .  
(c) Purchasing power rises because the real value exceeds EUR 10000.

**95. GDC**

- (a) A: EUR 9 380.67; B: EUR 9 375.73.  
(b) A gives the larger value.

**96.**

- (a)  $V = 2000(.7)(.8) = 1120$  euros.  
(b) Total percentage loss =  $100(1 - .56) = 44\%$ .  
(c) Percentages act on different starting values; losses cannot simply be added.

**97. GDC**

- (a) Effective rate =  $100[(1 + .042/12)^{12} - 1] = 4.28\%$ .  
(b)  $n > \ln 2 / \ln(1 + .042/12) = 198.3884$  months; first month 199.

**98. GDC**

- (a) After 12 years:  $80(1.028)^{12} = \text{EUR } 111.43$ .  
(b)  $n > \ln(1.5) / \ln(1.028) = 14.6827$ ; first year 15.

**99. GDC**

- (a)  $P = 50000/(1.008)^{40} = \text{EUR } 36\,353.71$ .  
(b) If the rate were 4% nominal quarterly:  $P = 50000/(1.01)^{40} = \text{EUR } 33\,582.66$ .

**100. GDC**

- (a) Annual multiplier =  $9600/12000 = 0.8$ ; depreciation 20%.  
(b)  $V_3 = 7680(.8) = 6144$  euros.  
(c)  $V_5 = 12000(.8)^5 = 3932.16$  euros.

**101. GDC**

- (a)  $(1 + r/200)^2 = 1.0609$ ;  $1 + r/200 = 1.03$ ;  $r = 6$ .  
(b) For EUR 1000 over two years:  $1000(1.03)^4 = 1125.51$  euros.

**102. GDC**

- (a) Real rate =  $100(1.03/1.04 - 1) = -0.962\%$ .  
(b) Real value after 5 years from EUR 6000: EUR 5 717.03.  
(c) The balance grows, but its purchasing power falls.

**1.4 | Financial mathematics (continued)****103.  GDC**

- (a) Value after 3 years: EUR 5 624.32.  
(b) Final value:  $5000(1.04)^3(1.05)^2 = \text{EUR } 6\,200.81$ .

**104.  GDC**

- (a)  $P(.9)^3 = 5832 \Rightarrow P = 8000$  euros.  
(b) After two more years:  $5832(.9)^2 = 4723.92$  euros.

**105.  GDC**

- (a) Annual growth factor  $= (1.004)^{12} = 1.04907021$ .  
(b) Real annual rate  $= 100[(1.004)^{12}/1.02 - 1] = 2.85\%$ .  
(c) Real value after 5 years  $= 10000(1.004)^{60}/(1.02)^5 = \text{EUR } 11\,508.58$ .

**1.5 | Integer exponents and introductory logarithms****Short response**

106.  $4^{-2} = \frac{1}{16}$ .  
107.  $(-3)^3 = -27$ .  
108.  $-3^4 = -81$ .  
109.  $(\frac{2}{3})^{-2} = \frac{9}{4}$ .  
110.  $x^{-3} = \frac{1}{x^3}$ .  
111.  $y^{-2-(-6)} = y^4$ .  
112.  $\log 0.0001 = -4$ .  
113.  $e^p = 7$ .  
114. 5 and  $-3$ , respectively.  
115. Neither has a real value: a real logarithm requires a strictly positive argument.

**Extended response**

116.  $\frac{4x^{-4}y^6}{4x^{-1}y} = x^{-3}y^5 = \frac{y^5}{x^3}$ .  
117.  $3x^{-2} = 3/x^2$ , while  $(3x)^{-2} = 1/(9x^2)$ . In the second, the exponent applies to 3 as well.  
118. Missing entries: 1000,  $-2$ , 1.  
119.  $t = \log 420$ , with  $2 < t < 3$  since  $100 < 420 < 1000$ .  
120.  $y = e^{-4} = 1/e^4 > 0$ . A negative logarithm corresponds to a positive argument below 1.  
121.  $3x - 6 > 0 \Rightarrow x > 2$ .  
122.  GDC  $\log 72 \approx 1.86$ .  
123.  GDC  $\ln 0.12 \approx -2.12$ .



## 1.5 | Integer exponents and introductory logarithms (continued)

**124.** **GDC**  $x = \log 570 \approx 2.76.$

**125.** **GDC**  $S = e^{1.8} \approx 6.05.$

**Exam-style**

**126.**

(a)  $p = -3, q = 4.$

(b)  $10^{p+q} = 10.$

(c)  $e^{pq} = e^{-12} = 1/e^{12}.$

**127.**

(a)  $E = a^{-2-(-3)}b^{4-1} = ab^3.$

(b)  $E = (-2)(3^3) = -54.$

**128.**

(a)  $e^p = 6.$

(b)  $e^{2p} = 36.$

(c)  $e^{-p} = 1/6.$

**129.**

(a)  $10^t = 250.$

(b)  $10^{t+1} = 2500.$

(c)  $10^{t-2} = 250/100 = 5/2.$

**130.**

(a)  $L = 10(5) = 50$  dB.

(b)  $\log R = 3 \Rightarrow R = 1000.$

(c)  $R = 1$ , so intensity equals the reference value, not zero.

**131.**

(a) pH = 6.

(b)  $c = 10^{-4}.$

(c) The second concentration is  $10^{-4}/10^{-6} = 100$  times the first.

**132.**

(a) At  $x = 2$ , left side  $1/4$  and right side  $-4$  differ.

(b)  $x^{-2} = 1/x^2.$

(c)  $(-x)^{-2} = 1/x^2 > 0.$

**133.** **GDC**

(a)  $x = \ln 0.04.$

(b)  $x = -3.22$  (3 s.f.).

(c) Because  $0 < 0.04 < 1$ , its natural logarithm is negative.



## 1.5 | Integer exponents and introductory logarithms (continued)

134.

(a)  $E = x^{2-3}y^{-4+1} = 1/(xy^3)$ .

(b) No: the original powers and denominator require  $x, y \neq 0$ .

135.

(a)  $8 - 2x > 0 \Rightarrow x < 4$ .

(b)  $8 - 2x = e^0 = 1 \Rightarrow x = 7/2$ , which satisfies the restriction.

## 1.7 | Rational exponents, logarithm laws and equations

## Short response

136.  $27^{2/3} = 9$ . Take the indicated root, then the integer power.

137.  $16^{-3/4} = \frac{1}{8}$ . Take the indicated root, then the integer power.

138.  $32^{3/5} = 8$ . Take the indicated root, then the integer power.

139.  $(\frac{9}{16})^{-1/2} = \frac{4}{3}$ . Take the indicated root, then the integer power.

140.  $x^{3/2-1/2} = x$ .

141.  $5 - 2 = 3$ .

142.  $\ln(x^2/y)$ .

143.  $2 + 2\log_3 x - \log_3 y$ .

144.  $\log_8 4 = \log_2 4 / \log_2 8 = 2/3$ .

145.  $x + 1 = 4 \Rightarrow x = 3$ .

## Extended response

146.  $8^{2/3} = 4$ ; expression  $= 4x^4/(2x) = 2x^3$ .

147.  $3^{2x-2} = 3^{6x+3}$ , so  $2x - 2 = 6x + 3$  and  $x = -5/4$ .

148.  $2x \ln 5 = \ln 7 \Rightarrow x = \frac{\ln 7}{2 \ln 5}$ .

149.  $e^{2x} = 7 \Rightarrow x = \frac{1}{2} \ln 7$ .

150.  $x - 1 = e^2 \Rightarrow x = 1 + e^2$ , with  $x > 1$ .

151. Domain  $x > 2$ .  $x(x - 2) = 8 \Rightarrow (x - 4)(x + 2) = 0$ ; only  $x = 4$  is valid.

152. Domain  $x > 0$ ;  $(x + 3)/x = 4 \Rightarrow x = 1$ .

153.  $\log 72 = 3p + 2q$ ;  $\log_3 8 = 3p/q$ .

154. Let  $t = 2^x > 0$ .  $t^2 - 5t + 4 = (t - 1)(t - 4) = 0$ , so  $x = 0$  or  $2$ .

155. Let  $t = e^x > 0$ .  $(t - 5)(t + 1) = 0$  gives  $t = 5$  only;  $x = \ln 5$ .

156. **GDC**  $\ln 20 / \ln 7 = 1.54$ .

157. **GDC**  $t = \ln(2.5) / \ln(1.08) = 11.9$ .



## 1.7 | Rational exponents, logarithm laws and equations (continued)

**158.**  $x - 2 > 0$  and  $7 - x > 0$ , so  $2 < x < 7$ .

**159.** Use  $x = y = 1$ :  $\log 2 \neq 0$ . Correctly,  $\log(xy) = \log x + \log y$ .

**160.**  $\ln(\sqrt{x+1}/(x-1)^2)$ , with original domain  $x > 1$ .

**Exam-style**

**161.**

(a)  $\log 40 = 3p + q$ .

(b)  $\log_4 25 = q/p$ .

(c)  $p + q = \log 10 = 1$ .

**162.**

(a)  $x > 1$ .

(b)  $\log_3(x^2 - 1) = 1 \Rightarrow x^2 - 1 = 3$ ; candidates  $\pm 2$ .

(c) Only  $x = 2$  satisfies  $x > 1$ .

**163.**

(a)  $t^2 - 10t + 16 = 0$ .

(b)  $(t - 2)(t - 8) = 0$ , so  $x = 1$  or  $3$ .

**164.**

(a)  $a^2 = 16$ , so  $a = 4$ .

(b)  $\log_a 8 = 3/2$ .

(c)  $\log_a(1/2) = -1/2$ .

**165.**  **GDC**

(a)  $N(0) = 120$ .

(b)  $t = \ln 3 / 0.3 = 3.66$  hours.

(c)  $N(5) = 120e^{1.5} = 538$ .

**166.**

(a)  $x > 0$ .

(b)  $(x + 6)/x = 4 \Rightarrow x = 2$ .

**167.**

(a)  $p = 81$ .

(b)  $\log_9 p = \log_3 p / \log_3 9 = 2$ .

(c)  $\log_p 3 = 1/4$ .

**168.**

(a)  $E = 9x^6 / (3x^{1/2}) = 3x^{11/2}$ .

(b) At  $x = 4$ ,  $E = 3 \cdot 2^{11} = 6144$ .

**1.7 | Rational exponents, logarithm laws and equations (continued)****169.**

- (a)  $x > 0$ .  
(b)  $x^2 + 2x - 8 = 0 \Rightarrow x = 2$  or  $-4$ ; only 2 is valid.

**170.**  **GDC**

- (a)  $t = \ln(1/2)/\ln(.92)$ .  
(b)  $t = 8.31$ .  
(c) First integer  $t$  below 40: 9.

**171.**

- (a) Right side  $= \frac{1}{2}(3 + \log_2 x)$ .  
(b)  $2\log_2 x = 3 + \log_2 x \Rightarrow x = 8$ .

**172.**

- (a)  $\log_2 45 = 2p + q$ .  
(b)  $\log_3(25/2) = (2q - 1)/p$ .

**173.**

- (a) Let  $t = e^x > 0$ :  $(t - 3)(t - 4) = 0$ .  
(b)  $x = \ln 3$  or  $\ln 4$ .

**174.**

- (a)  $1 < x < 5$ .  
(b)  $(x - 1)(5 - x) = 3 \Rightarrow x^2 - 6x + 8 = 0$ ;  $x = 2, 4$ , both valid.

**175.**

- (a) Correct left side  $= x + 1$ .  
(b)  $x + 1 = 2x \Rightarrow x = 1$ .  
(c) The equality holds at one value, so it is an equation, not an identity.

**1.8 | Infinite geometric series****Short response**

- 176.**  $|r| < 1$ ;  $S_\infty = a/(1 - r) = 12$ .  
**177.**  $|r| < 1$ ;  $S_\infty = a/(1 - r) = 12$ .  
**178.**  $|r| < 1$ ;  $S_\infty = a/(1 - r) = 4$ .  
**179.**  $|r| < 1$ ;  $S_\infty = a/(1 - r) = 36$ .  
**180.** No:  $r = 2$ , so  $|r| \geq 1$ .  
**181.** No:  $r = -1$ ; the partial sums alternate between 2 and 0 and do not approach one value.  
**182.**  $9/(1 - r) = 12 \Rightarrow r = 1/4$ .



## 1.8 | Infinite geometric series (continued)

**183.**  $a = 10(1 + 1/2) = 15.$

**184.**  $|x| < 1$ , equivalently  $-1 < x < 1.$

**185.**  $0.3 + 0.03 + \dots = (3/10)/(1 - 1/10) = 1/3.$

**Extended response**

**186.**

(a)  $S_1 = 4, S_2 = 6, S_3 = 7, S_4 = 15/2.$

(b)  $S_\infty = 8.$

(c)  $8 - S_n = 8(1/2)^n.$

**187.**

(a) First two rebound heights: 6 m and  $18/5$  m.

(b) Total =  $10 + 2[6/(1 - 3/5)] = 40$  m.

**188.**  $27/100 + (27/100)(1/100) + \dots = \frac{27/100}{1 - 1/100} = 3/11.$

**189.**

(a)  $r = 1 - 12/16 = 1/4.$

(b)  $S_3 = 16(1 - 1/64) = 63/4.$

**190.**  $S_\infty = 6$ ; remainder  $6/2^n < 1/10 \Rightarrow 2^n > 60$ . Least  $n = 6$ .

**191.**  $|2x - 1| < 1 \Rightarrow 0 < x < 1$ . Sum =  $5/[1 - (2x - 1)] = 5/(2 - 2x).$

**192.**

(a) Each new area is half the previous area; next areas 32, 16, 8.

(b) Sum =  $64/(1 - 1/2) = 128 \text{ cm}^2$ . This is a sum of areas counted separately, not the area of their union.

**193.**  $a = 18(1 - r)$  and  $ar = 4$ , so  $9r^2 - 9r + 2 = 0$ . Ratios  $1/3, 2/3$  give first terms 12, 6. Both satisfy  $|r| < 1$ .

**194.** First term  $1/8$ , ratio  $1/2$ ; sum =  $1/4$ .

**195.**  $S_\infty = 21/4$ ; absolute remainder =  $|S_\infty r^4| = 7/108.$

**Exam-style**

**196.**

(a)  $S_\infty = 50.$

(b)  $S_4 = 50(1 - 81/625) = 1088/25.$

(c) Remainder =  $50(3/5)^4 = 162/25.$

**197.**

(a)  $|x + 1| < 1 \Rightarrow -2 < x < 0.$

(b)  $S_\infty = -6/x.$

(c)  $-6/x = 12 \Rightarrow x = -1/2$ , inside the convergence interval.



## 1.8 | Infinite geometric series (continued)

**198.**

- (a)  $h_n = 12(1/2)^n$  for rebound  $n \geq 1$ .  
(b)  $12 + 2[6/(1 - 1/2)] = 36$  m.

**199.**

- (a)  $S_2 = S_\infty(1 - r^2)$ , so  $r^2 = 1/4$  and  $r = \pm 1/2$ .  
(b)  $a = 24(1 - r)$  gives  $a = 12$  or  $36$ .

**200.**  **GDC**

- (a)  $S_\infty = 25$ .  
(b)  $25(.8)^n < 0.01 \Rightarrow n > \ln(.0004)/\ln(.8) = 35.0628$ ; least  $n = 36$ .

**201.**

- (a)  $18/100 + 18/10000 + \dots$ .  
(b)  $\text{Sum} = (18/100)/(1 - 1/100) = 2/11$ .

**202.**

- (a)  $4/(1 - r) = 12 \Rightarrow r = 2/3$ .  
(b) Then  $4/(1 + r) = 12/5 \neq 6$ ; the stated data are inconsistent.

**203.**

- (a)  $a = 20(3/4) = 15$ .  
(b)  $S_3 = 20(1 - 1/64) = 315/16$ .  
(c) Remaining fraction  $= (1/4)^3 = 1/64$ .

**204.**

- (a)  $|x^2| < 1 \Rightarrow -1 < x < 1$ .  
(b)  $2/(1 - x^2) = 8/3 \Rightarrow x^2 = 1/4$ ;  $x = \pm 1/2$ .

**205.**

- (a)  $a = S_1 = 15$  and  $r = -1/2$ .  
(b)  $S_\infty = 10$ .  
(c)  $S_n - 10 = -10(-1/2)^n$ : negative for even  $n$ , positive for odd  $n$ .



## 1.9 | Binomial theorem: nonnegative integer powers

## Short response

206.  $\binom{5}{2} = 10$ . Use  $n!/[k!(n-k)!]$  and cancel factors.

207.  $\binom{6}{3} = 20$ . Use  $n!/[k!(n-k)!]$  and cancel factors.

208.  $\binom{8}{2} = 28$ . Use  $n!/[k!(n-k)!]$  and cancel factors.

209.  $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$ .

210.  $(1-x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$ .

211.  $(2+x)^3 = 8 + 12x + 6x^2 + x^3$ .

212.  $\binom{4}{2}3^2 = 54$ .

213.  $\binom{5}{3}2^2(-1)^3 = -40$ .

214.  $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ , for  $r = 0, 1, \dots, n$ .

215. The next row is 1, 5, 10, 10, 5, 1.

## Extended response

216.  $1 - 12x + 60x^2 - 160x^3 + \dots$

217.  $\binom{6}{4}2^23^4 = 4860$ .

218.  $T_{r+1} = \binom{4}{r}2^r x^{4-2r}$ . Set  $r = 2$ ; constant term 24.

219. Convolution:  $3(1) + 3(-4) + 1(4) = -5$ .

220.  $5k = 15 \Rightarrow k = 3$ . Quadratic coefficient  $= 10k^2 = 90$ .

221. There are six terms; the middle terms are  $10x^2$  and  $10x^3$ .

222. There are five terms; the third term is  $\binom{4}{2}(2x)^2 = 24x^2$ .

223.  $(100+1)^3 = 1000000 + 30000 + 300 + 1 = 1030301$ .

224.  $r = 2$  or  $5$ , using symmetry and  $\binom{7}{2} = 21$ .

225.  **GDC**  $\binom{30}{12} = 86493225$ .

## Exam-style

226.

(a)  $1 + 10x + 40x^2 + 80x^3 + \dots$

(b) Coefficient  $= \binom{5}{4}2^4 = 80$ .

227.

(a)  $6k^2 = 54 \Rightarrow k = 3$ .

(b) Coefficient of  $x^3 = 4k^3 = 108$ .



## 1.9 | Binomial theorem: nonnegative integer powers (continued)

**228.**

- (a)  $T_{r+1} = \binom{6}{r} 2^{6-r} (-1)^r x^{6-2r}$ .  
(b) Constant:  $r = 3$ , value  $20(8)(-1) = -160$ .  
(c)  $x^2$ :  $r = 2$ , coefficient  $15(16) = 240$ .

**229.**

- (a) Coefficient  $= 6 - 2(4) = -2$ .  
(b) Coefficient  $= 4 - 2(6) = -8$ .

**230.**

- (a)  $n = 8$ .  
(b) Coefficient of  $x^2 = \binom{8}{2} = 28$ .  
(c) Sum of coefficients is  $(1 + 1)^8 = 256$ .

**231.**

- (a)  $n(n-1)/2 = 15 \Rightarrow (n-6)(n+5) = 0$ ;  $n = 6$ .  
(b) Coefficient of  $x^3 = \binom{6}{3} = 20$ .

**232.**

- (a)  $81 - 108x + 54x^2 - 12x^3 + x^4$ .  
(b) Sum of coefficients: set  $x = 1$ , giving  $2^4 = 16$ .

**233.**

- (a)  $P(1) = 64$ ,  $P(-1) = 0$ .  
(b) Even-coefficient sum  $= (64 + 0)/2 = 32$ ; odd-coefficient sum also 32.

**234.**

- (a)  $T_{r+1} = \binom{6}{r} 2^r x^{12-3r}$ .  
(b) Constant:  $r = 4$ , coefficient  $15(16) = 240$ .  
(c) For  $x^3$ ,  $r = 3$ , coefficient  $20(8) = 160$ .

**235.**

- (a) Eight terms, so the middle terms are the fourth and fifth.  
(b) Fourth:  $\binom{7}{3} 2^4 x^3 = 560x^3$ ; fifth:  $\binom{7}{4} 2^3 x^4 = 280x^4$ .

**236.**

- (a)  $10k^3 = -80 \Rightarrow k = -2$ .  
(b) Coefficient of  $x^2 = 10k^2 = 40$ .

**237.**

- (a)  $1 - 12x + 54x^2 + \dots$ .  
(b)  $x = 0.01$  gives  $1 - 0.12 + 0.0054 = 0.8854$ .  
(c) Exact omitted contribution  $= -108x^3 + 81x^4 < 0$  at 0.01, so the estimate is an overestimate.

**1.9 | Binomial theorem: nonnegative integer powers (continued)****238.  GDC**

- (a)  $\binom{15}{8}2^8$ .  
(b) Coefficient = 1647360.

**239.**

- (a)  $3a + b = 5$ ,  $3a^2 + 3ab = 9$ .  
(b)  $b = 5 - 3a$  gives  $2a^2 - 5a + 3 = 0$ . Thus  $(a, b) = (1, 2)$  or  $(3/2, 1/2)$ .

**240.**

- (a)  $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ .  
(b) At  $x = 1$ , sum = 32; at  $x = -1$ , alternating sum = 0.  
(c) Equal sums of even and odd coefficients, each 16.

**AHL 1.10 | HL exclusive: binomial series with negative and rational exponents****Short response**

- 241.**  $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$ . Condition  $|x| < 1$ , so  $-1 < x < 1$ .  
**242.**  $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ . Condition  $|-x| < 1$ , so  $-1 < x < 1$ .  
**243.**  $(1+2x)^{-3} = 1 - 6x + 24x^2 - 80x^3 + \dots$ . Condition  $|2x| < 1$ , so  $-\frac{1}{2} < x < \frac{1}{2}$ .  
**244.**  $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$ . Condition  $|x| < 1$ , so  $-1 < x < 1$ .  
**245.**  $(1-2x)^{-\frac{1}{2}} = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \dots$ . Condition  $|-2x| < 1$ , so  $-\frac{1}{2} < x < \frac{1}{2}$ .  
**246.**  $(1-x)^{\frac{3}{2}} = 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \dots$ . Condition  $|-x| < 1$ , so  $-1 < x < 1$ .  
**247.**  $(1+3x)^{\frac{1}{3}} = 1 + x - x^2 + \frac{5}{3}x^3 + \dots$ . Condition  $|3x| < 1$ , so  $-\frac{1}{3} < x < \frac{1}{3}$ .  
**248.**  $(1-x)^{-\frac{1}{3}} = 1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \dots$ . Condition  $|-x| < 1$ , so  $-1 < x < 1$ .  
**249.**  $(1+x)^{\frac{2}{3}} = 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 + \dots$ . Condition  $|x| < 1$ , so  $-1 < x < 1$ .  
**250.**  $(1+\frac{1}{2}x)^{-2} = 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \dots$ . Condition  $|\frac{1}{2}x| < 1$ , so  $-2 < x < 2$ .  
**251.**  $(4+x)^{1/2} = 2(1+(\frac{1}{4})x)^{\frac{1}{2}} = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \dots$ ;  $-4 < x < 4$ .  
**252.**  $(9-3x)^{-1/2} = \frac{1}{3}(1+(-\frac{1}{3})x)^{-\frac{1}{2}} = \frac{1}{3} + \frac{1}{18}x + \frac{1}{72}x^2 + \dots$ ;  $-3 < x < 3$ .  
**253.**  $(8+x)^{1/3} = 2(1+(\frac{1}{8})x)^{\frac{1}{3}} = 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \dots$ ;  $-8 < x < 8$ .  
**254.**  $(16-8x)^{3/4} = 8(1+(-\frac{1}{2})x)^{\frac{3}{4}} = 8 - 3x - \frac{3}{16}x^2 + \dots$ ;  $-2 < x < 2$ .  
**255.** Coefficient =  $\frac{p(p-1)\cdots(p-3)}{4!}k^4 = 5$ , using  $p = -2$ ,  $k = 1$ .  
**256.** Coefficient =  $\frac{p(p-1)\cdots(p-3)}{4!}k^4 = -\frac{5}{8}$ , using  $p = \frac{1}{2}$ ,  $k = 2$ .  
**257.** Coefficient =  $\frac{p(p-1)\cdots(p-2)}{3!}k^3 = \frac{35}{16}$ , using  $p = -\frac{3}{2}$ ,  $k = -1$ .

**AHL 1.10 | HL exclusive: binomial series with negative and rational exponents (continued)**

**258.** Coefficient =  $\frac{p(p-1)\cdots(p-3)}{4!}k^4 = -\frac{10}{3}$ , using  $p = \frac{1}{3}$ ,  $k = 3$ .

**259.**  $|-4x| < 1 \Rightarrow -1/4 < x < 1/4$ . Since  $1/5 < 1/4$ , it lies inside.

**260.**  $|(x-1)/2| < 1 \Rightarrow -1 < x < 3$ .

**Extended response**

**261.** Use  $1 + \frac{1}{2}x - \frac{1}{8}x^2$  with  $x = \frac{1}{25}$ . The estimate is  $\frac{5099}{5000}$ . Here  $|x| < 1$ .

**262.** Use  $1 - \frac{1}{2}x + \frac{3}{8}x^2$  with  $x = -\frac{1}{25}$ . The estimate is  $\frac{5103}{5000}$ . Here  $|x| < 1$ .

**263.** Use  $1 + \frac{1}{3}x - \frac{1}{9}x^2$  with  $x = \frac{3}{50}$ . The estimate is  $\frac{2549}{2500}$ . Here  $|x| < 1$ .

**264.** Use  $1 - 2x + 3x^2$  with  $x = \frac{1}{10}$ . The estimate is  $\frac{83}{100}$ . Here  $|x| < 1$ .

**265.** Use  $2 + x - \frac{1}{4}x^2$  with  $x = -\frac{1}{100}$ . The estimate is  $\frac{79599}{40000}$ . Here  $|x| < 1$ .

**266. GDC**

(a)  $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$ .

(b)  $P_3(0.2) = 1.09550000$ ; true value = 1.09544512.

(c) Absolute error = 0.00005488. The approximation is above the true value.

**267. GDC**

(a)  $P_3(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$ .

(b)  $P_3(0.1) = 0.95343750$ ; true value = 0.95346259.

(c) Absolute error = 0.00002509. The approximation is below the true value.

**268. GDC**

(a)  $P_3(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$ .

(b)  $P_3(-0.15) = 0.94729167$ ; true value = 0.94726824.

(c) Absolute error = 0.00002343. The approximation is above the true value.

**269.**

(a)  $P_2$  is the quadratic binomial approximation; its coefficients are 1, 1/2, -1/8.

(b) The curves are closer near  $x = 0$  and separate farther away.

(c) No.  $|x| < 1$  guarantees series convergence, not high accuracy of a particular truncation.

**270.** Multiply the truncated series and collect powers:  $1 + 4x + 7x^2 + 10x^3 + \cdots$ ;  $-1 < x < 1$ .

**271.** Multiply the truncated series and collect powers:  $1 + \frac{3}{2}x + \frac{11}{8}x^2 + \frac{23}{16}x^3 + \cdots$ ;  $-1 < x < 1$ .

**272.** Multiply the truncated series and collect powers:  $1 + \frac{1}{2}x^2 + \cdots$ ;  $-1 < x < 1$ .

**273.** Multiply the truncated series and collect powers:  $1 - 4x + \frac{9}{2}x^2 - 7x^3 + \cdots$ ;  $-\frac{1}{2} < x < \frac{1}{2}$ .

**274.** Multiply the truncated series and collect powers:  $1 + 2x + 3x^2 + 3x^3 + \cdots$ ;  $-1 < x < 1$ .

**275.**  $2p = -3 \Rightarrow p = -3/2$ . Quadratic coefficient =  $p(p-1)2^2/2 = 15/2$ .

**276.**  $-2a = 6 \Rightarrow a = -3$ . Coefficients:  $x^2$ ,  $3a^2 = 27$ ;  $x^3$ ,  $-4a^3 = 108$ .

**AHL 1.10 | HL exclusive: binomial series with negative and rational exponents (continued)**

**277.**  $p(p-1)/2 = 3/8 \Rightarrow 4p^2 - 4p - 3 = 0$ . Thus  $p = 3/2$  or  $-1/2$ ; linear coefficient equals  $p$ .

**278.** Write  $2(1+x/4)^{1/2}$ . Correctly:  $2 + x/4 - x^2/64 + \dots$ . The inner variable is  $x/4$ , not  $x$ .

**279.** Condition  $|3x| < 1$ : yes at 0.2; no at 0.4; no at  $1/3$ , where the original expression is undefined. Do not infer convergence from the existence of a finite polynomial truncation.

**280.**  $\binom{-2}{r}(-1)^r = r + 1$ . The first five coefficients are 1, 2, 3, 4, 5.

**Exam-style**

**281.**

(a)  $f(x) = 1 + 2x + 4x^2 + 8x^3 + \dots$

(b)  $|-2x| < 1$ , so  $-\frac{1}{2} < x < \frac{1}{2}$ .

(c)  $P_3(\frac{1}{20}) = 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$ . Any equivalent exact sum is accepted.

**282.**

(a)  $f(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots$

(b)  $|2x| < 1$ , so  $-\frac{1}{2} < x < \frac{1}{2}$ .

(c)  $P_3(\frac{1}{50}) = 1 + \frac{1}{50} - \frac{1}{5000} + \frac{1}{250000}$ . Any equivalent exact sum is accepted.

**283.**

(a)  $f(x) = \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \frac{5}{2048}x^3 + \dots$

(b)  $|\frac{1}{4}x| < 1$ , so  $-4 < x < 4$ .

(c)  $P_3(\frac{1}{5}) = \frac{1}{2} + \frac{1}{80} + \frac{3}{6400} + \frac{1}{51200}$ . Any equivalent exact sum is accepted.

**284.**

(a)  $f(x) = 2 + \frac{1}{6}x - \frac{1}{72}x^2 + \frac{5}{2592}x^3 + \dots$

(b)  $|\frac{1}{4}x| < 1$ , so  $-4 < x < 4$ .

(c)  $P_3(\frac{1}{10}) = 2 + \frac{1}{60} - \frac{1}{7200} + \frac{1}{518400}$ . Any equivalent exact sum is accepted.

**285.**

(a)  $f(x) = 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \dots$

(b)  $|-x| < 1$ , so  $-1 < x < 1$ .

(c)  $P_3(\frac{1}{10}) = 1 - \frac{3}{20} + \frac{3}{800} + \frac{1}{16000}$ . Any equivalent exact sum is accepted.

**286.**

(a)  $pa = -2$  and  $p(p-1)a^2/2 = 6$ .

(b) Dividing by  $(pa)^2 = 4$  gives  $(p-1)/(2p) = 3/2$ , so  $p = -1/2$ ,  $a = 4$ .

(c) Cubic coefficient  $= p(p-1)(p-2)a^3/6 = -20$ ;  $|x| < 1/4$ .

**287.**

(a)  $f(x) = (1-x^2)^{-1/2}$  for  $|x| < 1$ .

(b)  $f(x) = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \dots$

(c) Only even powers occur because the substituted variable is  $x^2$  (also  $f(-x) = f(x)$ ).

**AHL 1.10 | HL exclusive: binomial series with negative and rational exponents (continued)****288.**

- (a)  $1 - 6x + 27x^2 - 108x^3 + \dots$ .  
(b) Coefficient of  $x^r$  is  $(-1)^r(r+1)3^r$ .  
(c) Coefficient of  $x^4$  is  $5 \cdot 81 = 405$ .

**289.**

- (a)  $(1-x)^{-1/2} = 1 + x/2 + 3x^2/8 + 5x^3/16 + \dots$ .  
(b)  $a + 1/2 = 0 \Rightarrow a = -1/2$ .  
(c)  $f(x) = 1 + x^2/8 + x^3/8 + \dots$ .

**290.**

- (a) First factor:  $1 - x - x^2/2 - x^3/2 + \dots$ ; second:  $1 - x + x^2 - x^3 + \dots$ .  
(b)  $f(x) = 1 - 2x + \frac{3}{2}x^2 - 2x^3 + \dots$ .  
(c) Need  $|2x| < 1$  and  $|x| < 1$ , so  $|x| < 1/2$ .

**291.**

- (a) From  $V \propto r^3$ ,  $r/r_0 = (1+x)^{1/3}$ .  
(b)  $r/r_0 = 1 + x/3 - x^2/9 + \dots$ .  
(c) For  $x = 0.06$ , fractional increase  $\approx 0.02 - 0.0004 = 0.0196$ , or 1.96%.

**292.**

- (a)  $R = (1+x)^{1/2}$ .  
(b)  $R \approx 1 + x/2 - x^2/8$ .  
(c) At  $x = 0.04$ ,  $R \approx 1.0198$ : increase 1.98%.

**293.  GDC**

- (a) At  $x = 0.1$ ,  $P_2 = 0.83$ ;  $f = 1/1.1^2 \approx 0.82644628$ .  
(b) Absolute error  $\approx 0.00355372$ ; percentage error  $\approx 0.430\%$ .  
(c) At  $x = 0.5$ ,  $P_2 = 0.75$  but  $f = 4/9$ ; finite truncation is much poorer farther from zero.

**294.**

- (a)  $p(p-1)/2 = p(p-1)(p-2)/6$ .  
(b) Cancel nonzero  $p(p-1)$ :  $3 = p-2$ , so  $p = 5$ .  
(c) Then the expansion terminates; equal coefficients are both 10. No negative or noninteger exponent satisfies these assumptions.

**295.**

- (a)  $f(x) = 1 + 2x^2 + 6x^4 + \dots$ .  
(b)  $|4x^2| < 1 \Rightarrow |x| < 1/2$ .  
(c) At  $x = 1/10$ , estimate  $= 1 + 1/50 + 3/5000 = 5103/5000$ .

**AHL 1.10 | HL exclusive: binomial series with negative and rational exponents (continued)****296.**

- (a)  $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ .
- (b)  $f(x) = 1 + 3x + 5x^2 + 7x^3 + \dots$ ; coefficient of  $x^r$  is  $2r + 1$ .
- (c) At  $x = 1/4$ , exact sum  $= (5/4)/(3/4)^2 = 20/9$ .

**297.**

- (a)  $3k^2/8 = 6 \Rightarrow k^2 = 16$ , so  $k = -4$ .
- (b) Expansion  $= 1 + 2x + 6x^2 + 20x^3 + \dots$ .
- (c) Guaranteed interval  $|x| < 1/4$ .

**298.**

- (a) Each expansion has coefficients  $1, \pm 1/2, -1/8, \pm 1/16, -5/128$ .
- (b)  $f(x) = 2 - x^2/4 - 5x^4/64 + \dots$ .
- (c) The odd powers cancel;  $f$  is even. Guaranteed interval  $|x| < 1$ .

**299.**

- (a)  $g(x) = x + x^3/8 + \dots$ .
- (b)  $g(1/5) \approx 1/5 + 1/1000 = 201/1000$ .
- (c) Even-power coefficients cancel because  $g(-x) = -g(x)$ .

**300.**

- (a)  $f(x) = \frac{1}{2}(1+x)^{-1/2}(1-2x)^{-1}$ .
- (b)  $f(x) = \frac{1}{2} + \frac{3}{4}x + \frac{27}{16}x^2 + \frac{103}{32}x^3 + \dots$ .
- (c) Intersection of  $|x| < 1$  and  $|2x| < 1$ :  $|x| < 1/2$ .
- (d) At  $x = 1/10$ , cubic estimate  $= \frac{1}{2} + \frac{3}{40} + \frac{27}{1600} + \frac{103}{32000} = 19043/32000$ .