



Section 1.15: Proof

IB Mathematics: Analysis and Approaches HL

Mathematical induction, proof by contradiction and counterexamples

50-question worksheet with original Paper 1-style practice

- Learning goal** Construct rigorous proofs by mathematical induction and contradiction, and disprove universal statements using a valid counterexample with a clear explanation.
- Syllabus link** IB Mathematics AA HL: **AHL 1.15**. Proof by mathematical induction; proof by contradiction; use of a counterexample to show that a statement is not always true. Induction may connect with complex numbers, differentiation, sums and divisibility.
- Structure** Section 1: 15 short response questions. Section 2: 15 extended response questions. Section 3: 20 original IB Paper 1-style questions.
- Calculator** This worksheet is intended as non-calculator practice. A calculator cannot replace the logical steps of a proof. Give exact arguments throughout.

Instructions

Show the structure of every proof clearly. For induction, include a **base case**, an **induction hypothesis**, the **inductive step** from k to $k + 1$, and a concluding statement. For contradiction, state the assumption being made and identify the contradiction. For a counterexample, do not merely state a value: explain why it satisfies the conditions of the claim but makes the conclusion false.

1. Short response questions (15 questions)

1. For each statement below, name the most natural proof method from *mathematical induction*, *contradiction*, or *counterexample*.

- (a) Prove a formula for $1 + 2 + \cdots + n$ for every $n \in \mathbb{Z}^+$.
- (b) Prove that $\sqrt{7}$ is irrational.
- (c) Show that the statement “the product of two irrational numbers is always irrational” is false.

2. Let $P(n)$ be the statement

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Verify the base case $P(1)$.

3. For the statement in Question 2, write down precisely what is assumed in the induction hypothesis $P(k)$, and write the statement that must be proved for $P(k+1)$.

4. In the inductive step for Question 2, simplify

$$\frac{k(k+1)}{2} + (k+1)$$

into the required form for $P(k+1)$.



5. Consider the claim that 4 divides $5^n - 1$ for every $n \in \mathbb{Z}^+$. Verify the base case $n = 1$.
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6. Give a counterexample to the statement “every prime number is odd”, and explain why it is a counterexample.
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7. Consider numbers of the form $n^2 + 41n + 41$, where $n \in \mathbb{N}$. Find a value of n showing that not every such number is prime, and explain your choice.
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8. Show that the statement “there are no positive integer solutions to $x^2 + y^2 = 10$ ” is false by giving and explaining a counterexample.
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9. To prove by contradiction that $\sqrt{2}$ is irrational, state the assumption that should be made at the start of the proof. Include the condition placed on the numerator and denominator.
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10. Let a be rational and b be irrational. To prove by contradiction that $a + b$ is irrational, suppose instead that $a + b = c$ is rational. Write the equation that produces the contradiction.
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11. A student verifies a formula for $n = 1, 2, 3, 4, 5$ and then claims it is true for every positive integer n . Explain why this is not a proof by induction.
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12. The claim $n^2 + n$ is even for every $n \in \mathbb{Z}^+$ is to be proved by induction. Assuming $k^2 + k$ is even, show that $(k + 1)^2 + (k + 1)$ is also even.
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13. Suppose $4^k - 1 = 3m$ for some integer m . Show that $4^{k+1} - 1$ is divisible by 3.
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14. State the most natural proof method for each claim.
- (a) There are infinitely many prime numbers.
 - (b) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{Z}^+$.
 - (c) The statement “if $x^2 = y^2$, then $x = y$ ” is not always true.
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15. Explain why, in a counterexample question, it is not sufficient simply to write down a number or pair of numbers.

**2. Extended response questions (15 questions)**

16. Prove by mathematical induction that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for every $n \in \mathbb{Z}^+$.

17. Prove by mathematical induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for every $n \in \mathbb{Z}^+$.

18. Prove by mathematical induction that 3 divides $4^n - 1$ for every $n \in \mathbb{Z}^+$.

19. Prove by mathematical induction that 7 divides $8^n - 1$ for every $n \in \mathbb{Z}^+$.

20. Prove by mathematical induction that $n^3 - n$ is divisible by 6 for every $n \in \mathbb{Z}^+$.

21. Prove by mathematical induction that

$$2^n \geq n + 1$$

for every integer $n \geq 0$.

22. Let $i^2 = -1$. Prove by mathematical induction that

$$(1 + i)^{2n} = 2^n i^n$$

for every $n \in \mathbb{Z}^+$.

23. Using the product rule in the inductive step, prove by mathematical induction that

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for every $n \in \mathbb{Z}^+$.

24. Prove by contradiction that $\sqrt{3}$ is irrational.

25. Prove by contradiction that $\sqrt[3]{5}$ is irrational.

26. Let r be a rational number and x an irrational number. Prove by contradiction that $r + x$ is irrational.

27. Prove by contradiction that there are infinitely many prime numbers. You may use the fact that every integer greater than 1 has a prime divisor.



28. Consider the statement

$$n^2 + n + 17 \text{ is prime for every } n \in \mathbb{N}.$$

Show that this statement is not always true. Your answer must explain why your example is a counterexample.

29. Show that the statement “if ab is even, then both a and b are even” is not always true, where $a, b \in \mathbb{Z}$.

30. A proof by induction begins with the claim that $4^n + 2$ is divisible by 3 for every $n \in \mathbb{Z}^+$.

- (a) Verify the base case.
- (b) Assume $4^k + 2 = 3m$ for some integer m . Show that $4^{k+1} + 2$ is divisible by 3.
- (c) Explain exactly where the induction hypothesis is used in your working.

**3. IB exam-style questions: Paper 1, non-calculator (20 questions)**

Paper 1-style practice. The questions in this section are original. They are modelled on the concise structure, command terms, exact reasoning and mark allocations used in IB Mathematics AA HL Paper 1 questions. Calculator use is not intended.

31. Let $P(n)$ be the statement

$$\sum_{r=1}^n (3r - 1) = \frac{n(3n + 1)}{2}, \quad n \in \mathbb{Z}^+.$$

- (a) Verify $P(1)$. [1 marks]
(b) Assume $P(k)$ is true. Prove that $P(k + 1)$ is true. [4 marks]
(c) Hence state the conclusion. [1 mark]

32. Prove by mathematical induction that

$$1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$$

for every $n \in \mathbb{Z}^+$. [6 marks]

33. Prove by mathematical induction that 7 divides $8^n - 1$ for every $n \in \mathbb{Z}^+$. [6 marks]

34. Prove by mathematical induction that 9 divides $10^n - 1$ for every $n \in \mathbb{Z}^+$. [6 marks]

35. Consider

$$S_n = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1).$$

Prove by mathematical induction that

$$S_n = \frac{n(n + 1)(n + 2)}{3}$$

for every $n \in \mathbb{Z}^+$. [7 marks]

36. Prove by mathematical induction that

$$n! \geq 2^{n-1}$$

for every $n \in \mathbb{Z}^+$. [6 marks]

37. Let $i^2 = -1$. Prove by mathematical induction that

$$(1 + i)^{4n} = (-4)^n$$

for every $n \in \mathbb{Z}^+$. [5 marks]

38. For real θ , let $\text{cis } \theta = \cos \theta + i \sin \theta$.

(a) Verify that $(\text{cis } \theta)^1 = \text{cis } \theta$. [1 mark]

(b) Assume $(\text{cis } \theta)^k = \text{cis}(k\theta)$. Using the compound-angle formulae, prove that

$$(\text{cis } \theta)^{k+1} = \text{cis}((k + 1)\theta).$$

[5 marks]

(c) Hence state De Moivre's theorem for positive integer powers. [1 mark]



39. Let $P(n)$ be the statement

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{Z}^+.$$

Using the product rule, prove $P(n)$ by mathematical induction.

[6 marks]

40. Prove by contradiction that $\sqrt{7}$ is irrational.

[6 marks]

41. Prove by contradiction that $\sqrt[3]{2}$ is irrational.

[6 marks]

42. Let $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$. Prove by contradiction that $a + b \notin \mathbb{Q}$.

[5 marks]

43.

(a) Prove by contradiction that $\sqrt{6}$ is irrational.

[4 marks]

(b) Hence prove that $\sqrt{2} + \sqrt{3}$ is irrational.

[4 marks]

44. Assume, for contradiction, that there are only finitely many primes $p_1, p_2, \dots, p_m$. Let

$$N = p_1 p_2 \cdots p_m + 1.$$

Use N to complete a proof that there are infinitely many prime numbers.

[7 marks]

45. Show that the statement

$$\text{“if } x^2 = y^2, \text{ then } x = y\text{”}$$

is not always true for real numbers x and y .

[3 marks]

46. Show that the statement “the product of two irrational numbers is irrational” is not always true.

[3 marks]

47. Let

$$P = \{n^2 + 41n + 41 : n \in \mathbb{N}\}.$$

Show that not every element of P is prime.

[4 marks]

48. Consider the statement

$$n^2 - n + 11 \text{ is prime for every } n \in \mathbb{Z}^+.$$

Show that this statement is not always true.

[3 marks]

49. Consider the claim $2^n > n^2$.

(a) Give a counterexample showing that the claim is not true for every $n \in \mathbb{Z}^+$.

[2 marks]

(b) Prove by mathematical induction that $2^n > n^2$ for every integer $n \geq 5$.

[6 marks]

50. Define $u_n = n^3 + 5n$ for $n \in \mathbb{Z}^+$.

(a) Explain why $k(k+1)$ is even for every integer k .

[2 marks]

(b) Prove by mathematical induction that u_n is divisible by 6 for every $n \in \mathbb{Z}^+$.

[6 marks]



Answer key and solution outlines

Section 1 answers: Questions 1–15

- (a) Mathematical induction. (b) Contradiction. (c) Counterexample.
- $1 = \frac{1(2)}{2} = 1$, so $P(1)$ is true.
- Assume $1 + 2 + \dots + k = \frac{k(k+1)}{2}$. Prove $1 + 2 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$.
- $\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$.
- $5^1 - 1 = 4$, which is divisible by 4.
- 2 is prime but is not odd, so it satisfies the hypothesis “prime” and violates the conclusion “odd”.
- Take $n = 40$: $40^2 + 41(40) + 41 = 1681 = 41^2$, which is composite.
- $(x, y) = (1, 3)$ gives $1^2 + 3^2 = 10$ with both values positive integers, contradicting the claim.
- Assume $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$, $\gcd(p, q) = 1$.
- $b = c - a$. Since $c, a \in \mathbb{Q}$, this would imply $b \in \mathbb{Q}$, contradiction.
- Checking finitely many cases provides evidence only; it does not establish the implication $P(k) \Rightarrow P(k+1)$.
- $(k+1)^2 + (k+1) = k^2 + 3k + 2 = (k^2 + k) + 2(k+1)$, a sum of two even integers.
- $4^{k+1} - 1 = 4(4^k - 1) + 3 = 12m + 3 = 3(4m + 1)$.
- (a) Contradiction. (b) Mathematical induction. (c) Counterexample.
- The example must be shown to satisfy the assumptions of the statement while making its conclusion false; otherwise it has not been established as a counterexample.

Section 2 answers: Questions 16–30

- Base $n = 1$: $1 = 1^2$. Assume $1 + 3 + \dots + (2k-1) = k^2$. Then adding the next odd number gives $k^2 + (2k+1) = (k+1)^2$. Therefore the formula holds for all $n \in \mathbb{Z}^+$.
- Base $n = 1$ is immediate. Assume $\sum_{r=1}^k r^2 = \frac{k(k+1)(2k+1)}{6}$. Then

$$\sum_{r=1}^{k+1} r^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6},$$

which is the required $k+1$ form.

- Base: $4 - 1 = 3$. Assume $4^k - 1 = 3m$. Then $4^{k+1} - 1 = 4(4^k - 1) + 3 = 3(4m + 1)$.
- Base: $8 - 1 = 7$. Assume $8^k - 1 = 7m$. Then $8^{k+1} - 1 = 8(8^k - 1) + 7 = 7(8m + 1)$.
- Base $n = 1$: 0 is divisible by 6. Assume $k^3 - k$ is divisible by 6. The increase to the next term is

$$[(k+1)^3 - (k+1)] - (k^3 - k) = 3k(k+1).$$

Since $k(k+1)$ is even, this increase is divisible by 6. Hence the result follows by induction.

- Base $n = 0$: $1 \geq 1$. Assume $2^k \geq k+1$. Then $2^{k+1} \geq 2(k+1) \geq k+2$ for $k \geq 0$.
- Base $n = 1$: $(1+i)^2 = 2i = 2^1 i^1$. Assume $(1+i)^{2k} = 2^k i^k$. Since $(1+i)^2 = 2i$,

$$(1+i)^{2(k+1)} = 2^k i^k (2i) = 2^{k+1} i^{k+1}.$$

- Base $n = 1$: $\frac{d}{dx} x = 1$. Assume $\frac{d}{dx} (x^k) = kx^{k-1}$. Using $x^{k+1} = x^k x$ and the product rule,

$$\frac{d}{dx} (x^{k+1}) = kx^{k-1} x + x^k = (k+1)x^k.$$

- Assume $\sqrt{3} = p/q$ in lowest terms. Then $p^2 = 3q^2$, so $3 \mid p$, say $p = 3r$. Hence $q^2 = 3r^2$, so $3 \mid q$, contradicting lowest terms.
- Assume $\sqrt[3]{5} = p/q$ in lowest terms. Then $p^3 = 5q^3$, so $5 \mid p$, say $p = 5r$. Thus $125r^3 = 5q^3$, so $q^3 = 25r^3$, hence $5 \mid q$, contradiction.
- Suppose $r + x = s \in \mathbb{Q}$. Then $x = s - r \in \mathbb{Q}$, contradicting that x is irrational.
- Suppose the primes are $p_1, \dots, p_m$. Let $N = p_1 \dots p_m + 1 > 1$. Any prime divisor q of N cannot be one of the p_i , since division by any p_i leaves remainder 1. This contradicts completeness of the list.



28. Take $n = 16$: $16^2 + 16 + 17 = 289 = 17^2$, so the value is not prime.
29. Take $a = 2$, $b = 3$. Then $ab = 6$ is even, but b is odd, so the conclusion fails.
30. (a) $4^1 + 2 = 6$, divisible by 3. (b) If $4^k + 2 = 3m$, then $4^{k+1} + 2 = 4(4^k + 2) - 6 = 12m - 6 = 3(4m - 2)$. (c) The assumption replaces $4^k + 2$ by $3m$.

Section 3 answers: Questions 31–40

31. (a) $2 = \frac{1(4)}{2}$. (b) Assume $\sum_{r=1}^k (3r - 1) = \frac{k(3k+1)}{2}$. Then

$$\sum_{r=1}^{k+1} (3r - 1) = \frac{k(3k+1)}{2} + (3k+2) = \frac{(k+1)(3k+4)}{2},$$

which is the $k+1$ form. (c) Hence true for all $n \in \mathbb{Z}^+$.

32. Base $n = 1$: $1 = (3 - 1)/2$. Assume $1 + 3 + \dots + 3^{k-1} = (3^k - 1)/2$. Add 3^k :

$$\frac{3^k - 1}{2} + 3^k = \frac{3^{k+1} - 1}{2}.$$

33. Base $8 - 1 = 7$. If $8^k - 1 = 7m$, then $8^{k+1} - 1 = 8(8^k - 1) + 7 = 7(8m + 1)$.
34. Base $10 - 1 = 9$. If $10^k - 1 = 9m$, then $10^{k+1} - 1 = 10(10^k - 1) + 9 = 9(10m + 1)$.
35. Base $n = 1$: $2 = 1 \cdot 2 \cdot 3/3$. Assume $S_k = \frac{k(k+1)(k+2)}{3}$. Then

$$S_{k+1} = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}.$$

36. Base $1! = 1 = 2^0$. Assume $k! \geq 2^{k-1}$. Then

$$(k+1)! = (k+1)k! \geq (k+1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k.$$

37. Base: $(1+i)^4 = -4$. Assume $(1+i)^{4k} = (-4)^k$. Then

$$(1+i)^{4(k+1)} = (1+i)^{4k}(1+i)^4 = (-4)^k(-4) = (-4)^{k+1}.$$

38. (a) Immediate. (b)

$$\begin{aligned} (\operatorname{cis} \theta)^{k+1} &= \operatorname{cis}(k\theta) \operatorname{cis} \theta \\ &= [\cos k\theta \cos \theta - \sin k\theta \sin \theta] + i[\sin k\theta \cos \theta + \cos k\theta \sin \theta] = \operatorname{cis}((k+1)\theta). \end{aligned}$$

(c) $(\operatorname{cis} \theta)^n = \operatorname{cis}(n\theta)$ for $n \in \mathbb{Z}^+$.

39. Base $n = 1$ is true. Assume $(x^k)' = kx^{k-1}$. Then

$$(x^{k+1})' = (x^k x)' = kx^{k-1}x + x^k = (k+1)x^k.$$

40. Assume $\sqrt{7} = p/q$ in lowest terms. Then $p^2 = 7q^2$, so $7 \mid p$; let $p = 7r$. This gives $q^2 = 7r^2$, so $7 \mid q$, contradiction.

Section 3 answers: Questions 41–50

41. Assume $\sqrt[3]{2} = p/q$ in lowest terms. Then $p^3 = 2q^3$, so $2 \mid p$; write $p = 2r$. Then $8r^3 = 2q^3$, so $q^3 = 4r^3$ and $2 \mid q$, contradiction.

42. Suppose $a + b = c \in \mathbb{Q}$. Then $b = c - a \in \mathbb{Q}$, contradicting $b \notin \mathbb{Q}$.

43. (a) Suppose $\sqrt{6} = p/q$ in lowest terms. Then $p^2 = 6q^2$, so both 2 and 3 divide p , hence $6 \mid p$. Writing $p = 6r$ gives $q^2 = 6r^2$, so $6 \mid q$, contradiction. (b) If $s = \sqrt{2} + \sqrt{3}$ were rational, then $s^2 = 5 + 2\sqrt{6}$, so $\sqrt{6} = (s^2 - 5)/2$ would be rational, contradicting part (a).

44. $N > 1$, so it has a prime divisor q . For each listed prime p_i , $N \equiv 1 \pmod{p_i}$, so no p_i divides N . Thus q is a prime not in the supposedly complete list, contradiction.

45. Take $x = 1$, $y = -1$. Then $x^2 = y^2 = 1$ but $x \neq y$.

46. $\sqrt{2}$ and $\sqrt{2}$ are irrational, but $\sqrt{2} \cdot \sqrt{2} = 2$ is rational.

47. With $n = 40$, $n^2 + 41n + 41 = 1681 = 41^2$, so this element of P is not prime.

48. Take $n = 11$: $11^2 - 11 + 11 = 121 = 11^2$, which is not prime.

49. (a) $n = 3$: $2^3 = 8 < 9 = 3^2$ (also $n = 4$ gives equality). (b) Base $n = 5$: $32 > 25$. Assume $2^k > k^2$ for $k \geq 5$. Then $2^{k+1} > 2k^2$. Also

$$2k^2 - (k+1)^2 = k^2 - 2k - 1 > 0 \quad (k \geq 5),$$

so $2^{k+1} > (k+1)^2$.



50. (a) Of two consecutive integers, one is even, so $k(k+1)$ is even. (b) Base $u_1 = 6$. Assume u_k is divisible by 6. Then

$$u_{k+1} - u_k = (k+1)^3 + 5(k+1) - (k^3 + 5k) = 3k(k+1) + 6.$$

By part (a), $3k(k+1)$ is divisible by 6, so the difference is divisible by 6. Hence u_{k+1} is divisible by 6.