



Advanced Integration

IB Mathematics AA HL | Worksheet 2

Powers of sine and cosine • Substitution • Integration by parts

Learning goal	Choose and combine integration methods, including suitable preparation of trigonometric powers.
Syllabus link	AHL 5.16: substitution, integration by parts and repeated integration by parts. Trigonometric identities support these methods.
Structure	15 short responses; 15 extended responses; 20 original IB exam-style questions.
Technology	Non-calculator throughout. Give exact answers. All angles are in radians.

Instructions

Work on separate paper. Show your substitution or your choices of u and dv . Change the limits when changing the variable in a definite integral. Include $+C$ for indefinite integrals. Give total area and distance as positive quantities. Less direct substitutions are supplied.

1 Short response

Questions 1–15

1. For each integral, state an efficient starting method and your first step. Do not evaluate.

(a) $\int \sin^3 x \cos^2 x \, dx$

(b) $\int \sin^2 x \cos^4 x \, dx$

(c) $\int x^3 \cos(x^2) \, dx$ (use $t = x^2$ first)

(d) $\int e^{3x} \sin(2x) \, dx$

2. Find $\int \sin^3 x \, dx$.

3. Find $\int \cos^5 x \, dx$.

4. Find $\int \sin^3 x \cos^4 x \, dx$.

5. Find $\int \sin^4 x \cos^3 x \, dx$.



1 Short response: method practice

Questions 6–15

Choosing a first step for $\int \sin^m x \cos^n x \, dx$

An odd sine power Save one $\sin x$; convert the remaining even sine power using $\sin^2 x = 1 - \cos^2 x$; use $u = \cos x$.

An odd cosine power Save one $\cos x$; convert the remaining even cosine power using $\cos^2 x = 1 - \sin^2 x$; use $u = \sin x$.

Both powers even Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. Repeat if needed. If both powers are odd, either substitution route is possible.

6. Find $\int \sin^2(3x) \, dx$.

7. Find $\int \cos^2(2x) \, dx$.

8. Find $\int \sin^2 x \cos^2 x \, dx$.

9. Find $\int \frac{4x}{(x^2 + 3)^2} \, dx$.

10. Find $\int \frac{e^x}{2 + e^x} \, dx$.

11. Find $\int \frac{(\ln x)^3}{x} \, dx$, where $x > 0$.

12. Find $\int x \sin(2x) \, dx$.

13. Find $\int x \ln(2x) \, dx$, where $x > 0$.

14. Find $\int x^2 e^{-2x} \, dx$.

15. A student writes $\int \sin^2 x \, dx = \frac{1}{3} \sin^3 x + C$. Explain the error by differentiating the proposed answer, then give a correct antiderivative.

**2 Extended response: trigonometric powers**

Questions 16–20

Show each identity before using it. For products, you may use $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$.

16. Guided example: both powers even. Consider $I = \int \sin^2 x \cos^4 x \, dx$.

- (a) Write the integrand as $(\sin x \cos x)^2 \cos^2 x$. Show that it equals $\frac{1}{16}(1 - \cos 4x)(1 + \cos 2x)$.
 - (b) Using $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$, express the integrand as a constant plus multiples of $\cos 2x$, $\cos 4x$ and $\cos 6x$.
 - (c) Hence find I .
 - (d) Evaluate $\int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$ exactly.
-

17. Find $\int \sin^4 x \cos^2 x \, dx$.

- (a) Use double-angle identities to rewrite the integrand without powers or products.
 - (b) Integrate and compare the signs of your sine terms with Question 16.
-

18. Consider $\int \sin^4 x \cos^4 x \, dx$.

- (a) Show that $\sin^4 x \cos^4 x = \frac{1}{128}(3 - 4 \cos 4x + \cos 8x)$.
 - (b) Hence find an antiderivative and evaluate the integral from 0 to $\pi/2$.
-

19. Find $\int \sin^5 x \cos^2 x \, dx$ using $u = \cos x$.

- (a) Show how the remaining even power of sine is rewritten.
 - (b) Find the antiderivative and evaluate the integral from 0 to $\pi/2$.
-

20. Evaluate $\int_0^{\pi/4} \sin^4(2x) \cos^3(2x) \, dx$.

- (a) Use $u = \sin 2x$ and state the transformed limits.
- (b) Evaluate the resulting polynomial integral.



2 Extended response: substitution and parts

Questions 21–25

21. Use $u = x^2 + 2$ to find $\int \frac{x^3}{(x^2 + 2)^3} dx$.

- (a) Rewrite both x^2 and $x dx$ in terms of u .
 - (b) Find the antiderivative, then evaluate the integral from 0 to $\sqrt{2}$.
-

22. Evaluate $\int_0^4 x\sqrt{2x+1} dx$ using $u = \sqrt{2x+1}$.

- (a) Express x , dx and the limits in terms of u .
 - (b) Evaluate the transformed integral.
-

23. Consider $\int \frac{\ln x}{x^2} dx$, where $x > 0$.

- (a) Use parts, differentiating $\ln x$, to find the integral.
 - (b) Hence evaluate $\int_1^e \frac{\ln x}{x^2} dx$.
-

24. A function F satisfies $F'(x) = \arcsin(2x)$ for $-\frac{1}{2} < x < \frac{1}{2}$, and $F(0) = 0$.

- (a) Use integration by parts to find $F(x)$.
 - (b) Find the exact value of $F(\frac{1}{4})$.
-

25. Use integration by parts twice to find $\int x^2 \sin(2x) dx$. Show both choices of u and dv , and check the result by differentiation.



2 Extended response: combining methods

Questions 26–30

26. Find $\int x^3 e^{2x} dx$ using repeated integration by parts. Hence evaluate $\int_0^1 x^3 e^{2x} dx$.

27. Let $I = \int e^{-x} \sin(2x) dx$ and $J = \int e^{-x} \cos(2x) dx$.

- (a) Use parts to obtain two equations linking I and J .
 - (b) Eliminate J and find I .
-

28. Evaluate $\int_0^1 x^3 \ln(1+x^2) dx$.

- (a) Use $u = 1 + x^2$ to show that the integral is $\frac{1}{2} \int_1^2 (u-1) \ln u du$.
 - (b) Use integration by parts to evaluate it exactly.
-

29. Use $u = 1 + e^x$ to find $\int e^{2x} \sqrt{1+e^x} dx$. Show explicitly how $e^{2x} dx$ is rewritten.

30. A student uses $u = 1 + x^2$ and writes

$$\int_0^{\sqrt{3}} \frac{x}{1+x^2} dx = \frac{1}{2} \int_0^{\sqrt{3}} \frac{1}{u} du.$$

- (a) Identify the error and write the correct transformed integral.
- (b) Evaluate the original integral exactly.



3 IB exam-style / Paper 1

Questions 31–34

Original exam-style questions. All questions are non-calculator; marks are shown in square brackets.

31.

[Maximum mark: 6]

Consider $I = \int_0^{\pi/3} \sin^3 x \cos^2 x \, dx$.

(a) Use $u = \cos x$ to express I as a polynomial integral with new limits. [3]

(b) Hence find the exact value of I . [3]

32.

[Maximum mark: 7]

Consider $\int_0^{\pi/4} \sin^4 x \, dx$.

(a) Show that $\sin^4 x = \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$. [3]

(b) Hence evaluate the integral exactly. [4]

33.

[Maximum mark: 7]

Let $f(x) = \sin^2(2x) \cos^4(2x)$.

(a) Show that $f(x) = \frac{1}{16} + \frac{1}{32} \cos 4x - \frac{1}{16} \cos 8x - \frac{1}{32} \cos 12x$. [4]

(b) Hence evaluate $\int_0^{\pi/4} f(x) \, dx$. [3]

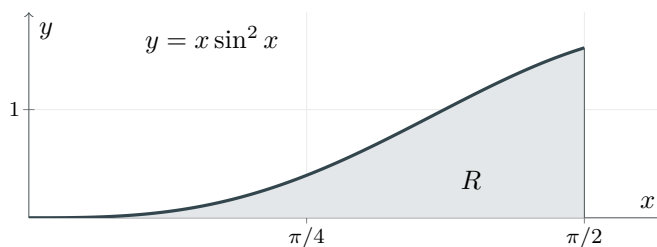
34.

[Maximum mark: 7]

The region R lies under $y = x \sin^2 x$ from $x = 0$ to $x = \pi/2$.

(a) Use a double-angle identity and integration by parts to find an antiderivative of $x \sin^2 x$. [5]

(b) Find the exact area of R . [2]





3 IB exam-style / Paper 1

Questions 35–38

35.

[Maximum mark: 6]

Let $I = \int_0^{\pi/2} x \sin^2 x \cos^2 x \, dx$.

- (a) Rewrite I using $\sin^2 x \cos^2 x = \frac{1}{8}(1 - \cos 4x)$. [1]
- (b) Use parts to find $\int x \cos 4x \, dx$. [3]
- (c) Hence evaluate I . [2]

36.

[Maximum mark: 7]

Consider the sixth power of sine.

- (a) Using $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and the product-to-sum identity, show that

$$\sin^6 x = \frac{1}{32}(10 - 15 \cos 2x + 6 \cos 4x - \cos 6x).$$

- (b) Find $\int \sin^6 x \, dx$. [3]
- (c) Hence evaluate $\int_0^{\pi/2} \sin^6 x \, dx$. [1]

37.

[Maximum mark: 8]

Let $I = \int_0^{\pi/2} \sin^5 x \cos^3 x \, dx$.

- (a) Choose a suitable substitution and express I as a polynomial integral. [2]
- (b) Find the exact value of I . [3]
- (c) Using $t = \pi/2 - x$, deduce the value of $\int_0^{\pi/2} \sin^3 x \cos^5 x \, dx$. Show the transformed limits. [3]

38.

[Maximum mark: 8]

Evaluate $I = \int_0^{3/2} \frac{x^2}{\sqrt{9-x^2}} \, dx$ using $x = 3 \sin \theta$, with $0 \leq \theta \leq \pi/6$.

- (a) Find dx and simplify $\sqrt{9-x^2}$, explaining your choice of sign. [3]
- (b) Show that $I = 9 \int_0^{\pi/6} \sin^2 \theta \, d\theta$. [2]
- (c) Hence evaluate I exactly. [3]



3 IB exam-style / Paper 1

Questions 39–42

39.

[Maximum mark: 7]

Let $I = \int_0^2 \frac{x^2}{(x^2 + 4)^2} dx$. Use $x = 2 \tan \theta$.

(a) Show that $I = \frac{1}{2} \int_0^{\pi/4} \sin^2 \theta d\theta$. [4]

(b) Hence evaluate I exactly. [3]

40.

[Maximum mark: 5]

Consider $I = \int_1^6 \frac{1}{\sqrt{x+3}(1+\sqrt{x+3})} dx$.

(a) Use $u = \sqrt{x+3}$ to express I as an integral in u , with new limits. [3]

(b) Hence find I in the form $a \ln b$. [2]

41.

[Maximum mark: 8]

Let $I = \int_0^{\ln 2} e^x \ln(1 + e^x) dx$.

(a) Use $u = 1 + e^x$ to transform I , including its limits. [3]

(b) Use integration by parts to find $\int \ln u du$. [3]

(c) Hence find the exact value of I . [2]

42.

[Maximum mark: 9]

Consider $I = \int_1^{e^2} \frac{(\ln x)^2}{\sqrt{x}} dx$.

(a) Using $u = \sqrt{x}$, show that $I = 8 \int_1^e (\ln u)^2 du$. [3]

(b) Use repeated integration by parts to find an antiderivative of $(\ln u)^2$. [4]

(c) Hence evaluate I . [2]



3 IB exam-style / Paper 1

Questions 43–46

43.

[Maximum mark: 9]

Let $I = \int_0^{1/2} x \arccos x \, dx$.

(a) Use integration by parts to show that $I = \frac{\pi}{24} + \frac{1}{2} \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} \, dx$. [4]

(b) Use $x = \sin \theta$ to evaluate the remaining integral. [3]

(c) Hence find I exactly. [2]

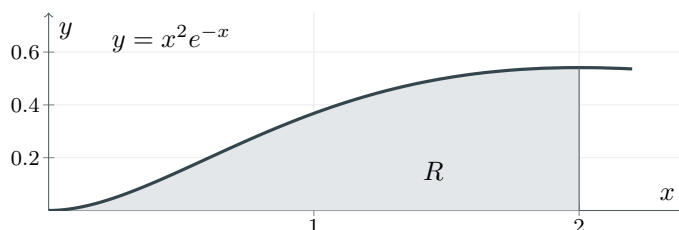
44.

[Maximum mark: 8]

The shaded region lies under $y = x^2 e^{-x}$ for $0 \leq x \leq 2$.

(a) Use integration by parts twice to find an antiderivative of $x^2 e^{-x}$. [5]

(b) Hence find the exact area of the shaded region. [3]



45.

[Maximum mark: 8]

Let $I = \int e^{3x} \cos 2x \, dx$.

(a) Apply integration by parts twice and obtain an equation in which I reappears. [5]

(b) Hence evaluate $\int_0^{\pi/4} e^{3x} \cos 2x \, dx$. [3]

46.

[Maximum mark: 9]

Consider $I = \int_0^{\pi/2} e^x \sin^2 x \, dx$.

(a) Rewrite I as a difference of two integrals using a double-angle identity. [1]

(b) Use integration by parts twice to find $\int e^x \cos 2x \, dx$. [5]

(c) Hence evaluate I exactly. [3]



3 IB exam-style / Paper 1

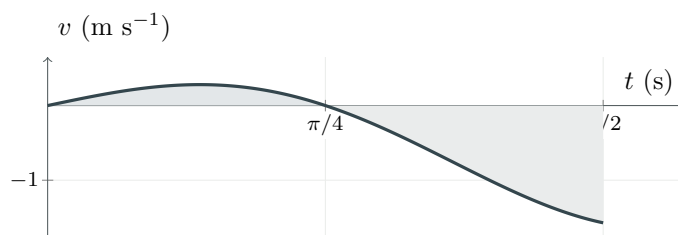
Questions 47–50

47.

[Maximum mark: 9]

A particle has velocity $v(t) = t \cos 2t \text{ m s}^{-1}$, for $0 \leq t \leq \pi/2$ seconds. Its initial displacement is zero.

- (a) Find the time in the interior of the interval at which the particle changes direction. [2]
- (b) Use integration by parts to find its displacement $s(t)$. [3]
- (c) Find its final displacement and its total distance travelled. [4]



48.

[Maximum mark: 8]

A function F satisfies $F'(x) = (ax^2 + b)e^x$, where a, b are constants. Also $F(0) = 0$, $F'(0) = 1$ and $F(1) = 3e - 5$.

- (a) Find b . [1]
- (b) Use repeated integration by parts to find $F(x)$ in terms of a and the integration constant. [4]
- (c) Hence find a and the equation of F . [3]

49.

[Maximum mark: 10]

For integers $n \geq 0$, define $I_n = \int_0^{\pi/2} \sin^n x \cos^2 x \, dx$.

- (a) Find I_0 . [2]
- (b) For $n \geq 2$, use integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x \cos^2 x \, dx$ to show that $I_n = \frac{n-1}{n+2} I_{n-2}$. [5]
- (c) Hence find I_4 and I_6 . [3]

50.

[Maximum mark: 11]

Consider $I = \int_1^{e^{\pi/2}} \cos^2(\ln x) \, dx$.

- (a) Using $u = \ln x$, express I entirely in terms of u , including the new limits. [3]
- (b) Use a double-angle identity and repeated integration by parts to find $\int e^u \cos^2 u \, du$. [6]
- (c) Hence evaluate I exactly. [2]



Answer key and solution outlines

Questions 1–15

Integration by parts: $\int u dv = uv - \int v du$. Use the boundary term $[uv]_a^b$ for definite integrals. Equivalent antiderivatives are accepted; they may differ by a constant.

1 Short response

1. (a) Save one $\sin x$; write $\sin^2 x = 1 - \cos^2 x$ and substitute $u = \cos x$. (b) Both powers are even: reduce powers with double-angle identities. (c) Substitute $t = x^2$, then use parts on $\frac{1}{2} \int t \cos t dt$. (d) Apply parts twice, then solve for the original integral.
2. Write $\sin^3 x = (1 - \cos^2 x) \sin x$. With $u = \cos x$, $du = -\sin x dx$, the result is $-\cos x + \frac{1}{3} \cos^3 x + C$.
3. Save $\cos x dx$ and use $u = \sin x$: $\int (1 - u^2)^2 du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$. Hence $\sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$.
4. With $u = \cos x$, the integral is $-\int (u^4 - u^6) du = -\frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x + C$.
5. With $u = \sin x$, the integral is $\int (u^4 - u^6) du = \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$.
6. Use $\sin^2(3x) = \frac{1}{2}(1 - \cos 6x)$: $\frac{x}{2} - \frac{\sin 6x}{12} + C$.
7. Use $\cos^2(2x) = \frac{1}{2}(1 + \cos 4x)$: $\frac{x}{2} + \frac{\sin 4x}{8} + C$.
8. Since $\sin^2 x \cos^2 x = \frac{1}{4} \sin^2(2x) = \frac{1}{8}(1 - \cos 4x)$, the result is $\frac{x}{8} - \frac{\sin 4x}{32} + C$.
9. $u = x^2 + 3$, $du = 2x dx$: $2 \int u^{-2} du = -\frac{2}{x^2+3} + C$.
10. $u = 2 + e^x$, $du = e^x dx$: $\int \frac{1}{u} du = \ln(2 + e^x) + C$. The logarithm argument is positive.
11. $u = \ln x$, $du = dx/x$: $\int u^3 du = \frac{1}{4}(\ln x)^4 + C$.
12. Parts with $u = x$, $dv = \sin 2x dx$, $v = -\frac{1}{2} \cos 2x$ gives $-\frac{x \cos 2x}{2} + \frac{1}{2} \int \cos 2x dx = -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} + C$.
13. Parts with $u = \ln(2x)$, $du = dx/x$, $v = x^2/2$ gives $\frac{x^2}{2} \ln(2x) - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln(2x) - \frac{x^2}{4} + C$.
14. Repeated parts, differentiating x^2 , gives $-\frac{1}{2}x^2e^{-2x} + \int xe^{-2x} dx = -e^{-2x} \left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4} \right) + C$.
15. The proposed derivative is $\sin^2 x \cos x$, which has an unwanted $\cos x$ factor. Instead use $\sin^2 x = (1 - \cos 2x)/2$: the correct answer is $\frac{x}{2} - \frac{\sin 2x}{4} + C$.



Answer key and solution outlines

Questions 16–18

2 Extended response

16.

(a) Use $\sin x \cos x = \frac{1}{2} \sin 2x$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$:

$$\sin^2 x \cos^4 x = \frac{1}{8} \sin^2(2x)(1 + \cos 2x) = \frac{1}{16}(1 - \cos 4x)(1 + \cos 2x).$$

(b) Expand first, then remove the remaining product:

$$\cos 4x \cos 2x = \frac{1}{2}(\cos 6x + \cos 2x).$$

Thus

$$\begin{aligned} \sin^2 x \cos^4 x &= \frac{1}{16}(1 + \cos 2x - \cos 4x - \cos 4x \cos 2x) \\ &= \frac{1}{16} + \frac{1}{32} \cos 2x - \frac{1}{16} \cos 4x - \frac{1}{32} \cos 6x. \end{aligned}$$

(c) Use $\int \cos(kx) dx = \sin(kx)/k + C$ term by term:

$$I = \frac{x}{16} + \frac{\sin 2x}{64} - \frac{\sin 4x}{64} - \frac{\sin 6x}{192} + C.$$

(d) At both endpoints all three sine terms are zero. Hence

$$\int_0^{\pi/2} \sin^2 x \cos^4 x dx = \left[\frac{x}{16} + \frac{\sin 2x}{64} - \frac{\sin 4x}{64} - \frac{\sin 6x}{192} \right]_0^{\pi/2} = \boxed{\frac{\pi}{32}}.$$

17.

(a) Write the integrand as $\frac{1}{16}(1 - \cos 4x)(1 - \cos 2x)$. After using the product-to-sum identity, this is $\frac{1}{16} - \frac{1}{32} \cos 2x - \frac{1}{16} \cos 4x + \frac{1}{32} \cos 6x$.(b) Integrating gives $\frac{x}{16} - \frac{\sin 2x}{64} - \frac{\sin 4x}{64} + \frac{\sin 6x}{192} + C$. Compared with Question 16, the $\sin 2x$ and $\sin 6x$ coefficients change sign.

18.

(a) $\sin^4 x \cos^4 x = \frac{1}{16} \sin^4(2x) = \frac{1}{64}(1 - 2\cos 4x + \cos^2 4x)$. Substitute $\cos^2 4x = (1 + \cos 8x)/2$ to obtain the stated identity.(b) An antiderivative is $\frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$. The definite integral is $\frac{3\pi}{256}$.



Answer key and solution outlines

Questions 19–24

2 Extended response

19.

(a) Save one $\sin x$ and write $\sin^4 x = (1 - \cos^2 x)^2$. With $u = \cos x$, $du = -\sin x dx$, the integral becomes $-\int(u^2 - 2u^4 + u^6) du$.

(b) The antiderivative is $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$. The definite integral is $0 - (-\frac{1}{3} + \frac{2}{5} - \frac{1}{7}) = \frac{8}{105}$.

20.

(a) $du = 2\cos 2x dx$ and $\cos^2 2x = 1 - u^2$. The limits become 0 and 1, so $I = \frac{1}{2}\int_0^1(u^4 - u^6) du$.

(b) $I = \frac{1}{2}[\frac{1}{5}u^5 - \frac{1}{7}u^7]_0^1 = \frac{1}{35}$.

21.

(a) $x^2 = u - 2$ and $x dx = \frac{1}{2} du$.

(b) $I = \frac{1}{2}\int(u^{-2} - 2u^{-3}) du = -\frac{1}{2u} + \frac{1}{2u^2} + C$. Replace u by $x^2 + 2$. For the definite integral, $u = 2$ to 4, giving $(-\frac{1}{8} + \frac{1}{32}) - (-\frac{1}{4} + \frac{1}{8}) = \frac{1}{32}$.

22.

(a) $x = (u^2 - 1)/2$, $dx = u du$, and $u = 1$ to 3.

(b) $I = \frac{1}{2}\int_1^3(u^4 - u^2) du = [\frac{1}{10}u^5 - \frac{1}{6}u^3]_1^3 = \frac{99}{5} + \frac{1}{15} = \frac{298}{15}$.

23.

(a) Take $u = \ln x$, $dv = x^{-2} dx$, so $du = dx/x$ and $v = -1/x$. Parts gives $-\frac{\ln x}{x} + \int x^{-2} dx = -\frac{\ln x + 1}{x} + C$.

(b) The value is $[-(\ln x + 1)/x]_1^e = 1 - \frac{2}{e}$.

24.

(a) Take $u = \arcsin(2x)$ and $dv = dx$. Then $du = 2 dx/\sqrt{1 - 4x^2}$ and

$$F = x \arcsin(2x) - \int \frac{2x}{\sqrt{1 - 4x^2}} dx = x \arcsin(2x) + \frac{1}{2}\sqrt{1 - 4x^2} + C.$$

The remaining integral uses $w = 1 - 4x^2$, $dw = -8x dx$. Since $F(0) = 0$, $C = -\frac{1}{2}$.

(b) $F(\frac{1}{4}) = \frac{\pi}{24} + \frac{\sqrt{3}}{4} - \frac{1}{2}$.



Answer key and solution outlines

Questions 25–30

2 Extended response

25. First take $u = x^2$, $dv = \sin 2x \, dx$, so $v = -\frac{1}{2} \cos 2x$:

$$I = -\frac{x^2 \cos 2x}{2} + \int x \cos 2x \, dx.$$

For the second integral take $u = x$, $dv = \cos 2x \, dx$, giving $\frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$. Hence

$$I = -\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C.$$

Differentiation cancels $-x \cos 2x + x \cos 2x$ and $\frac{1}{2} \sin 2x - \frac{1}{2} \sin 2x$, leaving $x^2 \sin 2x$.

26. Repeatedly differentiate the polynomial and integrate e^{2x} :

$$\begin{aligned}\int x^3 e^{2x} \, dx &= \frac{1}{2} x^3 e^{2x} - \frac{3}{2} \int x^2 e^{2x} \, dx, \\ \int x^2 e^{2x} \, dx &= \frac{1}{2} x^2 e^{2x} - \int x e^{2x} \, dx, \\ \int x e^{2x} \, dx &= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C.\end{aligned}$$

Combining gives $e^{2x}(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}) + C$. At 1 this is $e^2/8$, and at 0 it is $-3/8$, so the definite integral is $(e^2 + 3)/8$.

27.

(a) Integrate the exponential and differentiate the trigonometric factor. Up to constants,

$$I = -e^{-x} \sin 2x + 2J, \quad J = -e^{-x} \cos 2x - 2I.$$

(b) Substitution gives $I = -e^{-x} \sin 2x - 2e^{-x} \cos 2x - 4I$. Thus $5I = -e^{-x}(\sin 2x + 2 \cos 2x)$ and

$$I = -\frac{e^{-x}}{5}(\sin 2x + 2 \cos 2x) + C.$$

28.

(a) $x^2 = u - 1$, $x \, dx = du/2$, and the limits become 1, 2.

(b) Parts gives $\int u \ln u \, du = \frac{1}{2} u^2 \ln u - \frac{1}{4} u^2$ and $\int \ln u \, du = u \ln u - u$, up to constants. Hence

$$I = \left[\left(\frac{u^2}{4} - \frac{u}{2} \right) \ln u - \frac{u^2}{8} + \frac{u}{2} \right]_1^2 = \frac{1}{2} - \frac{3}{8} = \frac{1}{8}.$$

29. $du = e^x \, dx$ and $e^x = u - 1$, so $e^{2x} \, dx = (u - 1) \, du$. Therefore

$$I = \int (u^{3/2} - u^{1/2}) \, du = \frac{2}{5} (1 + e^x)^{5/2} - \frac{2}{3} (1 + e^x)^{3/2} + C.$$

30.

(a) The limits are still x -values. When $x = 0$, $u = 1$; when $x = \sqrt{3}$, $u = 4$. The correct integral is $\frac{1}{2} \int_1^4 \frac{1}{u} \, du$.

(b) $\frac{1}{2} [\ln u]_1^4 = \frac{1}{2} \ln 4 = \ln 2$.



Answer key and solution outlines

Questions 31–34

3 IB exam-style

31.

(a) $\sin^3 x \cos^2 x \, dx = (1 - u^2)u^2(-du)$, and u runs from 1 to $\frac{1}{2}$. Reverse the limits: $I = \int_{1/2}^1 (u^2 - u^4) \, du$.

(b) $I = [u^3/3 - u^5/5]_{1/2}^1 = \frac{2}{15} - (\frac{1}{24} - \frac{1}{160}) = \frac{47}{480}$.

32.

(a) Square $\sin^2 x = (1 - \cos 2x)/2$ and replace $\cos^2 2x$ by $(1 + \cos 4x)/2$.

(b) The antiderivative is $\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$. At $x = \pi/4$, this is $\frac{3\pi}{32} - \frac{1}{4}$; at 0 it is zero. Hence $I = \frac{3\pi}{32} - \frac{1}{4}$.

33.

(a) Use $\sin 2x \cos 2x = \frac{1}{2} \sin 4x$, then reduce the remaining squares. This gives $\frac{1}{16}(1 - \cos 8x)(1 + \cos 4x)$. With $\cos 8x \cos 4x = \frac{1}{2}(\cos 12x + \cos 4x)$, the result follows.

(b) Integrate to get $[\frac{x}{16} + \frac{\sin 4x}{128} - \frac{\sin 8x}{128} - \frac{\sin 12x}{384}]_0^{\pi/4} = \frac{\pi}{64}$.

34.

(a) $\int x \sin^2 x \, dx = \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx$. Parts with $u = x$, $v = \frac{1}{2} \sin 2x$ gives $\int x \cos 2x \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$. Thus an antiderivative is $H(x) = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}$.

(b) The integrand is non-negative. Area $= H(\pi/2) - H(0) = \frac{\pi^2}{16} + \frac{1}{8} - (-\frac{1}{8}) = \frac{\pi^2}{16} + \frac{1}{4}$.



Answer key and solution outlines

Questions 35–38

3 IB exam-style

35.

(a) $I = \frac{1}{8} \int_0^{\pi/2} x \, dx - \frac{1}{8} \int_0^{\pi/2} x \cos 4x \, dx.$

(b) Parts with $u = x$, $v = \frac{1}{4} \sin 4x$ gives $\frac{x \sin 4x}{4} + \frac{\cos 4x}{16} + C.$

(c) The second definite integral is $1/16 - 1/16 = 0$, so $I = \frac{1}{8} [x^2/2]_0^{\pi/2} = \frac{\pi^2}{64}.$

36.

(a) Set $c = \cos 2x$. Then $\sin^6 x = (1 - 3c + 3c^2 - c^3)/8$. Use $c^2 = (1 + \cos 4x)/2$ and $c^3 = (3 \cos 2x + \cos 6x)/4$, the latter obtained by product-to-sum. Collecting terms gives the stated identity.

(b) $\frac{5x}{16} - \frac{15 \sin 2x}{64} + \frac{3 \sin 4x}{64} - \frac{\sin 6x}{192} + C.$

(c) All sine terms vanish at the endpoints, leaving $\frac{5\pi}{32}.$

37.

(a) Save $\cos x \, dx$ and use $u = \sin x$: $I = \int_0^1 (u^5 - u^7) \, du.$

(b) $I = [u^6/6 - u^8/8]_0^1 = \frac{1}{24}.$

(c) With $t = \pi/2 - x$, $dx = -dt$, the limits change from $0, \pi/2$ to $\pi/2, 0$. The integral becomes $-\int_{\pi/2}^0 \cos^3 t \sin^5 t \, dt = I = \frac{1}{24}.$

38.

(a) $dx = 3 \cos \theta \, d\theta$ and $\sqrt{9 - x^2} = 3\sqrt{1 - \sin^2 \theta} = 3 \cos \theta$, because $\cos \theta > 0$ on the stated interval.

(b) Substitution cancels the factors $3 \cos \theta$, leaving $9 \sin^2 \theta \, d\theta$. The limits are 0 and $\arcsin(1/2) = \pi/6$.

(c) $I = 9[\theta/2 - \sin 2\theta/4]_0^{\pi/6} = \frac{3\pi}{4} - \frac{9\sqrt{3}}{8}.$



Answer key and solution outlines

Questions 39–42

3 IB exam-style

39.

(a) $dx = 2 \sec^2 \theta d\theta$ and $x^2 + 4 = 4 \sec^2 \theta$. Thus $\frac{x^2}{(x^2+4)^2} dx = \frac{1}{2} \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = \frac{1}{2} \sin^2 \theta d\theta$. The bounds become 0 and $\pi/4$.

(b) $I = [\theta/4 - \sin 2\theta/8]_0^{\pi/4} = \frac{\pi}{16} - \frac{1}{8}$.

40.

(a) $x = u^2 - 3$, $dx = 2u du$, and the bounds are $u = 2, 3$. Therefore $I = 2 \int_2^3 \frac{1}{1+u} du$.

(b) $I = 2[\ln(1+u)]_2^3 = 2 \ln(4/3)$.

41.

(a) $du = e^x dx$; the limits are 2, 3, so $I = \int_2^3 \ln u du$.

(b) Write $\ln u = 1 \cdot \ln u$. Parts gives $u \ln u - \int 1 du = u \ln u - u + C$.

(c) $I = [u \ln u - u]_2^3 = 3 \ln 3 - 2 \ln 2 - 1$.

42.

(a) $x = u^2$, $dx = 2u du$ and $\ln x = 2 \ln u$. Thus $(\ln x)^2 dx / \sqrt{x} = 8(\ln u)^2 du$; the bounds become 1, e .

(b) Parts first gives $u(\ln u)^2 - 2 \int \ln u du$. A second application gives $u[(\ln u)^2 - 2 \ln u + 2] + C$.

(c) The expression in part (b) is e at $u = e$ and 2 at $u = 1$. Hence $I = 8(e - 2)$.



Answer key and solution outlines

Questions 43–46

3 IB exam-style

43.

(a) Take $u = \arccos x$ and $dv = x dx$, so $du = -dx/\sqrt{1-x^2}$ and $v = x^2/2$. The boundary term is $[x^2 \arccos x/2]_0^{1/2} = \pi/24$, giving the stated expression.

(b) With $x = \sin \theta$, $dx = \cos \theta d\theta$ and $\theta = 0$ to $\pi/6$, the integral is $\int_0^{\pi/6} \sin^2 \theta d\theta = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$.

(c) $I = \frac{\pi}{24} + \frac{1}{2}(\frac{\pi}{12} - \frac{\sqrt{3}}{8}) = \frac{\pi}{12} - \frac{\sqrt{3}}{16}$.

44.

(a) Parts gives $\int x^2 e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx$. Another application gives $\int x e^{-x} dx = -(x+1)e^{-x} + C$, so an antiderivative is $H(x) = -(x^2 + 2x + 2)e^{-x}$.

(b) The integrand is non-negative. Area $= H(2) - H(0) = -10e^{-2} - (-2) = 2 - \frac{10}{e^2}$.

45.

(a) Put $J = \int e^{3x} \sin 2x dx$. Up to constants,

$$I = \frac{1}{3}e^{3x} \cos 2x + \frac{2}{3}J, \quad J = \frac{1}{3}e^{3x} \sin 2x - \frac{2}{3}I.$$

Thus $I = \frac{1}{3}e^{3x} \cos 2x + \frac{2}{9}e^{3x} \sin 2x - \frac{4}{9}I$, so $I = \frac{e^{3x}}{13}(3 \cos 2x + 2 \sin 2x) + C$.

(b) Evaluate at $\pi/4$ and 0: the result is $(2e^{3\pi/4} - 3)/13$.

46.

(a) $I = \frac{1}{2} \int_0^{\pi/2} e^x dx - \frac{1}{2} \int_0^{\pi/2} e^x \cos 2x dx$.

(b) With $K = \int e^x \cos 2x dx$, $L = \int e^x \sin 2x dx$, parts gives $K = e^x \cos 2x + 2L$ and $L = e^x \sin 2x - 2K$, up to constants. Hence $K = \frac{e^x}{5}(\cos 2x + 2 \sin 2x) + C$.

(c) An antiderivative of $e^x \sin^2 x$ is $\frac{e^x}{2} - \frac{e^x}{10}(\cos 2x + 2 \sin 2x)$. Its endpoint values are $3e^{\pi/2}/5$ and $2/5$, so $I = (3e^{\pi/2} - 2)/5$.



Answer key and solution outlines

Questions 47–50

3 IB exam-style

47.

- (a) For $t > 0$, the sign is the sign of $\cos 2t$. This changes from positive to negative at $t = \pi/4$ s.
- (b) Parts with $u = t$, $dv = \cos 2t \, dt$ gives $s = \frac{t \sin 2t}{2} + \frac{\cos 2t}{4} + C$. Since $s(0) = 0$, $C = -\frac{1}{4}$.
- (c) $s(\pi/2) = -\frac{1}{2}$ m. At the turn, $s(\pi/4) = \frac{\pi}{8} - \frac{1}{4}$. Total distance is

$$\left(\frac{\pi}{8} - \frac{1}{4}\right) + \left[\left(\frac{\pi}{8} - \frac{1}{4}\right) - \left(-\frac{1}{2}\right)\right] = \frac{\pi}{4} \text{ m.}$$

The negative final displacement is a signed position change, not a negative distance.

48.

- (a) $F'(0) = b = 1$.
- (b) Two applications of parts give $\int x^2 e^x \, dx = e^x(x^2 - 2x + 2) + C$. Thus $F = e^x[a(x^2 - 2x + 2) + 1] + C$.
- (c) $F(0) = 0$ gives $C = -2a - 1$. Then $(a + 1)e - 2a - 1 = 3e - 5$, so $a(e - 2) = 2(e - 2)$ and $a = 2$. Hence $F(x) = (2x^2 - 4x + 5)e^x - 5$.

49.

- (a) $I_0 = \int_0^{\pi/2} \cos^2 x \, dx = [x/2 + \sin 2x/4]_0^{\pi/2} = \pi/4$.
- (b) Substitution gives $v = -\frac{1}{3} \cos^3 x$. For $n \geq 2$, the boundary term $[-\frac{1}{3} \sin^{n-1} x \cos^3 x]_0^{\pi/2}$ is zero. Parts therefore gives

$$I_n = \frac{n-1}{3} \int_0^{\pi/2} \sin^{n-2} x \cos^4 x \, dx = \frac{n-1}{3} (I_{n-2} - I_n),$$

using $\cos^4 x = \cos^2 x(1 - \sin^2 x)$. Rearranging: $(n+2)I_n = (n-1)I_{n-2}$.

- (c) $I_2 = \frac{1}{4}I_0 = \pi/16$; $I_4 = \frac{3}{6}I_2 = \pi/32$; $I_6 = \frac{5}{8}I_4 = 5\pi/256$.

50.

- (a) $x = e^u$ and $dx = e^u \, du$. The limits are $u = 0, \pi/2$, so $I = \int_0^{\pi/2} e^u \cos^2 u \, du$.
- (b) Use $\cos^2 u = (1 + \cos 2u)/2$. For $K = \int e^u \cos 2u \, du$ and $L = \int e^u \sin 2u \, du$, parts gives $K = e^u \cos 2u + 2L$, $L = e^u \sin 2u - 2K$, up to constants. Thus $K = e^u(\cos 2u + 2 \sin 2u)/5 + C$. Consequently

$$\int e^u \cos^2 u \, du = \frac{e^u}{2} + \frac{e^u}{10}(\cos 2u + 2 \sin 2u) + C.$$

- (c) At $\pi/2$, the antiderivative is $2e^{\pi/2}/5$; at 0 it is $3/5$. Hence $I = (2e^{\pi/2} - 3)/5$.

Source alignment. AHL 5.16 in the supplied IB Mathematics: analysis and approaches guide. Question style informed by the supplied AA HL Paper 1 calculus collection; all questions here are original. This is a separate second worksheet; the first worksheet remains available.