

Introduction to integration

IB Mathematics: Analysis and Approaches HL

60 progressive questions • Foundations and first applications

Learning goal	Reverse differentiation, find constants and connect integrals with area.
Syllabus link	SL 5.5; selected introductory skills from SL 5.10 and SL 5.11, all included in AA HL.
Structure	Q1–30: antiderivatives; Q31–50: definite integrals and area; Q51–60: optional next steps.
Technology	All questions can be completed without a calculator.

Instructions. Show your working on separate paper. Give exact answers and include $+C$ for indefinite integrals. Check antiderivatives by differentiation. In radical questions, take $x > 0$. All angles are in radians.
Choose your starting point. Begin with Q1–30. Attempt Q31–50 once definite integrals have been introduced. Q51–60 are optional if those functions and reverse-chain-rule ideas have been taught.

1 Reverse differentiation and the constant

Q1–10

- Find $\frac{d}{dx}(x^3)$. Hence write down $\int 3x^2 dx$.
- Write down two different functions whose derivative is $2x$. Explain why both are possible.
- Evaluate $\int 7 dx$.
- Evaluate $\int x dx$.
- Evaluate $\int x^2 dx$.
- Evaluate $\int 4x^3 dx$.
- Evaluate $\int -6x^2 dx$.
- Evaluate $\int (3x^2 + 4x - 5) dx$.
- A student writes $\int x^4 dx = 4x^3 + C$. Identify the mistake and give the correct answer.
- Use differentiation to check whether $F(x) = 2x^3 - 3x + 8$ is an antiderivative of $6x^2 - 3$.

2 The power rule and algebra**Q11–20**

Rewrite or expand before integrating where helpful. The power rule applies to x^n when $n \neq -1$.

11. Evaluate $\int (x^5 - 2x^2 + 3) dx$.

12. Evaluate $\int \frac{5}{x^2} dx$. Write the integrand as $5x^{-2}$ first.

13. Evaluate $\int \frac{6}{x^3} dx$. Write the integrand as a power of x first.

14. Evaluate $\int \sqrt{x} dx$. Use $\sqrt{x} = x^{1/2}$.

15. Evaluate $\int 3\sqrt[3]{x^2} dx$. Use $\sqrt[3]{x^2} = x^{2/3}$.

16. Evaluate $\int \frac{4}{\sqrt{x}} dx$. Write the integrand using a negative power.

17. Evaluate $\int (x + 2)(x - 1) dx$. Expand first.

18. Evaluate $\int (2x - 3)^2 dx$. Expand first.

19. Evaluate $\int \frac{x^3 + 2x^2 - 4}{x^2} dx$. Divide each term by x^2 first.

20. Evaluate $\int \frac{\sqrt{x} + 1}{\sqrt{x}} dx$. Simplify the fraction first.

3 Finding the constant from a given point

Q21–30

Integrate first, then substitute the given coordinates or function value.

21. Given $\frac{dy}{dx} = 4x$ and $y = 3$ when $x = 0$, find y .

22. Given $f'(x) = 3x^2 - 2$ and $f(1) = 4$, find $f(x)$.

23. Given $\frac{dy}{dx} = 6x - 4$ and $y = 7$ when $x = 2$, find y .

24. Given $f'(x) = 2x + 3$ and $f(-1) = 5$, find $f(x)$.

25. Given $\frac{dy}{dx} = 12x^2 - 6x + 2$ and $y = 4$ when $x = 1$, find y .

26. Given $f'(x) = \frac{4}{x^2}$ and $f(1) = 1$ and $x > 0$, find $f(x)$.

27. Given $f'(x) = 3\sqrt{x}$ and $f(4) = 21$ and $x > 0$, find $f(x)$.

28. Two functions satisfy $f'(x) = g'(x) = 2x$, with $f(1) = 5$ and $g(1) = -2$.

(a) Find $f(x)$ and $g(x)$.

(b) Find $f(x) - g(x)$ and interpret this graphically.

29. A curve has gradient function $3x^2 + 2x$ and passes through $(0, -1)$. Find its equation.

30. Given $f'(x) = kx$, $f(0) = 1$ and $f(2) = 7$:

(a) find the constant k ;

(b) find $f(1)$.

4 Definite integrals**Q31–40**

Use an antiderivative F and evaluate $F(b) - F(a)$, where appropriate.

31. Evaluate $\int_0^2 3x^2 dx$.

32. Evaluate $\int_1^3 (2x + 1) dx$.

33. Evaluate $\int_0^1 (4x^3 - 2x) dx$.

34. Evaluate $\int_{-1}^2 x^2 dx$.

35. Evaluate $\int_1^4 \frac{1}{x^2} dx$.

36. Evaluate $\int_1^4 \sqrt{x} dx$.

37. Evaluate $\int_0^2 3x^2 dx$ using $F(x) = x^3 + C$. Show explicitly why the constant C does not affect the answer.

38. Given $\int_1^4 f(x) dx = 7$, find:

(a) $\int_4^1 f(x) dx$;

(b) $\int_1^1 f(x) dx$.

39. Given $\int_1^3 f(x) dx = 5$ and $\int_3^5 f(x) dx = -2$, find $\int_1^5 f(x) dx$.

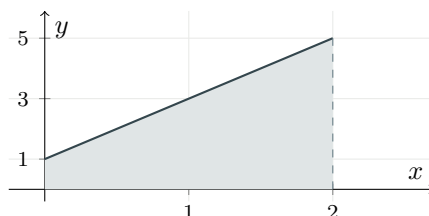
40. Find k if $\int_0^2 kx dx = 12$.

5 Area and accumulated change

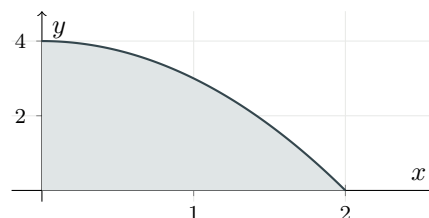
Q41–45

Write an integral before calculating each area. Area is always non-negative.

41. The shaded region lies under $y = 2x + 1$, above the x -axis, between $x = 0$ and $x = 2$. Write down a definite integral for its area and evaluate it.



42. Find the shaded area bounded by $y = 4 - x^2$, the positive coordinate axes. Start by writing a definite integral.



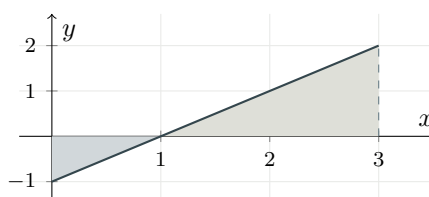
43. Find the area under $y = x^2$ between $x = 1$ and $x = 3$, above the x -axis.

44. For $y = x - 2$ on $0 \leq x \leq 2$:

- evaluate $\int_0^2 (x - 2) dx$;
- find the area between the graph and the x -axis. Explain the difference between your answers.

45. The graph of $y = x - 1$ is shown for $0 \leq x \leq 3$.

- State where the graph crosses the x -axis.
- Evaluate $\int_0^3 (x - 1) dx$.
- Find the total area between the graph and the x -axis on this interval.



5 Area and accumulated change (continued)

Q46–50

Connect the integral with the region or change described.

46. The graph of $y = f(x)$ consists of two straight line segments joining $(0, 0)$, $(2, 4)$ and $(4, 0)$.

- Use geometry to find $\int_0^4 f(x) dx$.
- Explain why the integral equals the area of the triangle.



47. A continuous function is non-positive on $[0, 2]$ and non-negative on $[2, 5]$. Given $\int_0^2 f(x) dx = -3$ and $\int_2^5 f(x) dx = 7$:

- find $\int_0^5 f(x) dx$;
- find the total area between its graph and the x -axis from $x = 0$ to $x = 5$.

48. The area under $y = x^2$, above the x -axis, from $x = 0$ to $x = a$ is 9 square units, where $a > 0$. Write an equation involving an integral and find a .

49. Water flows into a tank at a rate $R(t) = 2t + 3$ litres per minute, where $0 \leq t \leq 4$. The tank initially contains 10 litres and no water leaves.

- Use an integral to find the volume added in the first 4 minutes.
- Find the volume in the tank after 4 minutes.

50. Given $f'(x) = 6x - 2$ and $f(1) = 3$:

- use a definite integral to find $f(3) - f(1)$;
- hence find $f(3)$.

6 Optional next steps

Q51–60

Attempt this section only if these standard integrals have been introduced. Q56–59 use inspection (reverse chain rule); check each result by differentiation.

51. Evaluate $\int \frac{1}{x} dx$.

52. Evaluate $\int e^x dx$.

53. Evaluate $\int \cos x dx$.

54. Evaluate $\int \sin x dx$.

55. Evaluate $\int \left(\frac{2}{x} + 3e^x - 4 \cos x \right) dx$.

56. Evaluate $\int (2x + 1)^3 dx$. Hint: differentiate $(2x + 1)^4$, then adjust the multiplier.

57. Evaluate $\int e^{3x} dx$. Hint: differentiate e^{3x} , then adjust the multiplier.

58. Evaluate $\int \cos(2x) dx$. Hint: differentiate $\sin(2x)$, then adjust the multiplier.

59. Evaluate $\int 2x(x^2 + 1)^3 dx$. Hint: differentiate $(x^2 + 1)^4$, then adjust the multiplier.

60. Given $f'(x) = e^x + \frac{1}{x}$, $x > 0$, and $f(1) = e$:

- (a) find $f(x)$;
- (b) differentiate your answer to check it.

Answer key and solution outlines

Equivalent algebraic forms are accepted. Each C denotes an arbitrary constant.

1 | Reverse differentiation and the constant

1. $3x^2$; hence $\int 3x^2 dx = x^3 + C$.
2. For example, x^2 and $x^2 + 5$. Constants differentiate to zero; the general answer is $x^2 + C$.
3. $7x + C$.
4. $\frac{x^2}{2} + C$.
5. $\frac{x^3}{3} + C$.
6. $x^4 + C$.
7. $-2x^3 + C$.
8. $x^3 + 2x^2 - 5x + C$.
9. They differentiated instead of integrating. The answer is $\frac{x^5}{5} + C$.
10. $f'(x) = 6x^2 - 3$, so it is an antiderivative.

2 | The power rule and algebra

11. $\frac{x^6}{6} - \frac{2x^3}{3} + 3x + C$.
12. Integrand: $5x^{-2}$. Integral: $-\frac{5}{x} + C$.
13. Integrand: $6x^{-3}$. Integral: $-\frac{3}{x^2} + C$.
14. Integrand: $x^{1/2}$. Integral: $\frac{2}{3}x^{3/2} + C$.
15. Integrand: $3x^{2/3}$. Integral: $\frac{9}{5}x^{5/3} + C$.
16. Integrand: $4x^{-1/2}$. Integral: $8\sqrt{x} + C$.
17. Integrand: $x^2 + x - 2$. Integral: $\frac{x^3}{3} + \frac{x^2}{2} - 2x + C$.
18. Integrand: $4x^2 - 12x + 9$. Integral: $\frac{4}{3}x^3 - 6x^2 + 9x + C$.
19. Integrand: $x + 2 - 4x^{-2}$. Integral: $\frac{x^2}{2} + 2x + \frac{4}{x} + C$.
20. Integrand: $1 + x^{-1/2}$. Integral: $x + 2\sqrt{x} + C$.

Answer key and solution outlines (continued)

3 | Finding the constant from a given point

21. $y = 2x^2 + C$; $3 = C$. Thus $y = 2x^2 + 3$.
22. $f(x) = x^3 - 2x + C$; $4 = 1 - 2 + C$. Thus $f(x) = x^3 - 2x + 5$.
23. $y = 3x^2 - 4x + C$; $7 = 12 - 8 + C$. Thus $y = 3x^2 - 4x + 3$.
24. $f(x) = x^2 + 3x + C$; $5 = 1 - 3 + C$. Thus $f(x) = x^2 + 3x + 7$.
25. $y = 4x^3 - 3x^2 + 2x + C$; $4 = 4 - 3 + 2 + C$. Thus $y = 4x^3 - 3x^2 + 2x + 1$.
26. $f(x) = -4/x + C$; $1 = -4 + C$. Thus $f(x) = 5 - \frac{4}{x}$.
27. $f(x) = 2x^{3/2} + C$; $21 = 16 + C$. Thus $f(x) = 2x^{3/2} + 5$.
28. (a) $f = x^2 + C_1$, $5 = 1 + C_1$, so $f(x) = x^2 + 4$. Similarly, $g(x) = x^2 - 3$. (b) $f - g = 7$: the graph of f is 7 units above that of g .
29. $y = \int (3x^2 + 2x) dx = x^3 + x^2 + C$. At $(0, -1)$, $C = -1$, so $y = x^3 + x^2 - 1$.
30. $f(x) = \frac{kx^2}{2} + C$. From $f(0) = 1$, $C = 1$; then $7 = 2k + 1$, so (a) $k = 3$; (b) $f(1) = \frac{5}{2}$.

4 | Definite integrals

31. $[x^3]_0^2 = 8$.
32. $[x^2 + x]_1^3 = 12 - 2 = 10$.
33. $[x^4 - x^2]_0^1 = 0$.
34. $[x^3/3]_{-1}^2 = 8/3 - (-1/3) = 3$.
35. $[-1/x]_1^4 = -1/4 - (-1) = 3/4$.
36. $[\frac{2}{3}x^{3/2}]_1^4 = \frac{2}{3}(8 - 1) = \frac{14}{3}$.
37. $F(2) - F(0) = (8 + C) - (0 + C) = 8$. The two copies of C cancel.
38. (a) -7 : reversing the limits reverses the sign. (b) 0 : the interval has zero width.
39. Add the integrals over adjacent intervals: $5 + (-2) = 3$.
40. $[kx^2/2]_0^2 = 2k = 12$, so $k = 6$.

Answer key and solution outlines (continued)

5 | Area and accumulated change

41. $A = \int_0^2 (2x + 1) dx = [x^2 + x]_0^2 = 6$ square units.
42. $A = \int_0^2 (4 - x^2) dx = [4x - x^3/3]_0^2 = 8 - 8/3 = 16/3$ square units.
43. $A = \int_1^3 x^2 dx = [x^3/3]_1^3 = 26/3$ square units.
44. (a) $[x^2/2 - 2x]_0^2 = -2$. (b) Area $= -\int_0^2 (x - 2) dx = 2$ square units. The graph is below the axis, so the integral is negative; area is positive.
45. (a) $x = 1$. (b) $[x^2/2 - x]_0^3 = 3/2$. (c) $A = -\int_0^1 (x - 1) dx + \int_1^3 (x - 1) dx = 1/2 + 2 = 5/2$ square units.
46. (a) $\frac{1}{2} \times 4 \times 4 = 8$. (b) $f(x) \geq 0$ throughout $[0, 4]$, so its integral equals the area under its graph.
47. (a) $-3 + 7 = 4$. (b) $3 + 7 = 10$ square units: use positive areas for both regions.
48. $\int_0^a x^2 dx = 9$, so $a^3/3 = 9$, $a^3 = 27$ and $a = 3$.
49. (a) Added volume $= \int_0^4 (2t + 3) dt = [t^2 + 3t]_0^4 = 28$ litres. (b) $10 + 28 = 38$ litres.
50. (a) $f(3) - f(1) = \int_1^3 (6x - 2) dx = [3x^2 - 2x]_1^3 = 21 - 1 = 20$. (b) $f(3) = 3 + 20 = 23$.

6 | Optional next steps

51. $\ln |x| + C$.
52. $e^x + C$.
53. $\sin x + C$.
54. $-\cos x + C$.
55. $2 \ln |x| + 3e^x - 4 \sin x + C$.
56. $\frac{(2x+1)^4}{8} + C$. Since $\frac{d}{dx}(2x+1)^4 = 8(2x+1)^3$, divide by 8.
57. $\frac{1}{3}e^{3x} + C$. Since $\frac{d}{dx}e^{3x} = 3e^{3x}$, divide by 3.
58. $\frac{1}{2}\sin(2x) + C$. Since $\frac{d}{dx}\sin(2x) = 2\cos(2x)$, divide by 2.
59. $\frac{(x^2+1)^4}{4} + C$. Since $\frac{d}{dx}(x^2+1)^4 = 8x(x^2+1)^3$, divide by 4.
60. (a) $f(x) = e^x + \ln x + C$. Since $e = e + \ln 1 + C$, $C = 0$, so $f(x) = e^x + \ln x$. (b) Differentiating gives $e^x + 1/x$, as required.

Syllabus scope. Based on the supplied *Mathematics: analysis and approaches guide*, first assessment 2021, printed pp. 63, 65–66. This is an introductory selection from SL 5.5, SL 5.10 and SL 5.11, which are also part of AA HL. It does not attempt full coverage of these sections or the later AHL integration techniques. All questions are original practice questions.