

# Kinematics & substitution

IB Mathematics: Analysis and Approaches HL

Changing variables in motion problems and integrals

|                      |                                                                                                                                                  |
|----------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Learning goal</b> | Choose a useful variable, set up the correct integral and interpret the result.                                                                  |
| <b>Syllabus link</b> | SL 5.9 (kinematics), SL 5.10–5.11 (substitution and definite integrals); AHL 5.16 (further substitution), with AHL 5.18 separation of variables. |
| <b>Structure</b>     | Two supplied questions; 16 further questions; worked methods and detailed solutions.                                                             |
| <b>Technology</b>    | Work analytically. A calculator is optional for checking decimal values and graphs.                                                              |

**Instructions.** Show the substitution and the transformed differential. Give exact answers unless otherwise stated. Use radians. Include units in motion problems. Work on separate paper.

**60-minute route** 0–8 min: variable choice below; 8–28 min: Q1–2; 28–40 min: SL method and Q7, Q11; 40–55 min: HL method and Q13, Q16; 55–60 min: recap. If SL substitution needs more time, use Q8–10 instead of the HL pair. Remaining questions provide follow-up practice.

**Quick contents:** motion methods p. 1; kinematics questions p. 2; SL substitution p. 3; HL substitution p. 4; substitution practice pp. 5–6; solutions from p. 7.

## 1 Motion: identify the independent variable

With displacement  $s$  in metres and time  $t$  in seconds,

$$v = \frac{ds}{dt}, \quad a = \frac{dv}{dt}.$$

If velocity is given as  $v(s)$ , use the chain rule:

$$a = \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds}.$$

This identity is **derived** from the definitions;  $dv/ds$  alone is not acceleration.

| Given      | Wanted                  | Useful relationship                             |
|------------|-------------------------|-------------------------------------------------|
| $v = v(s)$ | Time between positions  | $\Delta t = \int_{s_1}^{s_2} \frac{1}{v(s)} ds$ |
| $a = a(v)$ | Displacement            | $\Delta s = \int_{v_0}^{v_1} \frac{v}{a(v)} dv$ |
| $v = v(t)$ | Displacement / distance | $\int v(t) dt \quad / \quad \int  v(t)  dt$     |

The first two forms follow by separating variables, on intervals where the indicated divisions are valid. A zero of  $v$  requires a check: it may be a turning point or only a momentary rest.

## 2 Kinematics questions

**1. Supplied question 1: velocity as a function of displacement.** A particle moves along a straight line. When its displacement from a fixed point  $O$  is  $s$  metres, its velocity is  $v(s) = 1/s$  metres per second.

- (a) Find its acceleration when  $s = 2$ .
- (b) Find the time taken to move from  $s = 2$  to  $s = 4$ .

*Here the stated positions are interpreted as positive displacements.*

**2. Supplied question 2: acceleration as a function of velocity.** A particle moves along a straight line with acceleration  $a = -(1 + v)$ , where  $v$  is its velocity. Initially  $v = 1 \text{ m s}^{-1}$ . Find its displacement from its starting point when it first comes to rest. Use SI units throughout.

**3. Choose the derivative.** Find the acceleration at the stated instant or position. Explain why the two answers differ.

- (a)  $v(t) = 3t^2 + 2$ , at  $t = 2$ .
- (b)  $v(s) = 3s^2 + 2$ , at  $s = 2$ .

**4. Sister question to Q1.** A particle has velocity  $v(s) = 4/s$ , for  $s > 0$ .

- (a) Find its acceleration when  $s = 2$ .
- (b) Find the time taken to move from  $s = 2$  to  $s = 6$ .
- (c) Take  $t = 0$  when  $s = 2$ . Find  $s(t)$  for  $t \geq 0$ .

**5. Sister question to Q2.** A particle has acceleration  $a = -(2 + v)$  and initial velocity  $4 \text{ m s}^{-1}$ .

- (a) Find the displacement up to its first rest.
- (b) Find the time taken to first come to rest.
- (c) Explain why the distance travelled equals the displacement over this interval.

**6. Displacement is not always distance.** A particle has velocity  $v(t) = t^2 - 4t + 3$  for  $0 \leq t \leq 4$ , and  $s(0) = 0$ .

- (a) Find the times at which it is at rest, and determine the direction of motion on each intervening interval.
- (b) Find its displacement and total distance travelled from  $t = 0$  to  $t = 4$ .
- (c) State when its speed is increasing, for  $0 < t < 4$ .

**Before integrating:** write down the variable you are integrating with respect to. A formula containing  $s$  or  $v$  cannot simply be integrated as though that variable were  $t$ .

### 3 SL substitution: undoing the chain rule

**The idea.** Differentiating  $F(g(x))$  gives  $F'(g(x))g'(x)$ . In an integral, look for an inner expression and its derivative, possibly multiplied by a constant.

**Method** 1. Choose  $u = g(x)$ . 2. Find  $du = g'(x)dx$ . 3. Rewrite the *whole* integral using  $u$  and  $du$ . 4. Integrate. 5. Substitute back, or evaluate using transformed limits. 6. Differentiate to check an indefinite answer.

#### Worked example A — matching a derivative up to a constant

$$\text{Find } \int 6x(x^2 + 4)^3 dx.$$

Choose  $u = x^2 + 4$ . Then  $du = 2xdx$ , so  $6xdx = 3du$ .

$$\int 6x(x^2 + 4)^3 dx = 3 \int u^3 du = \frac{3}{4}u^4 + C = \boxed{\frac{3}{4}(x^2 + 4)^4 + C}.$$

**Check:**  $\frac{3}{4} \cdot 4(x^2 + 4)^3 \cdot 2x = 6x(x^2 + 4)^3$ .

#### Worked example B — a logarithm

$$\text{Find } \int \frac{2x}{x^2 + 5} dx.$$

Let  $u = x^2 + 5$ , so  $du = 2xdx$ . Therefore

$$\int \frac{2x}{x^2 + 5} dx = \int \frac{1}{u} du = \ln |u| + C = \boxed{\ln(x^2 + 5) + C}.$$

Here  $x^2 + 5 > 0$ . In general, retain absolute-value signs for  $\ln |u|$ .

#### Worked example C — definite limits must change too

$$\text{Evaluate } \int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2 + 1}} dx.$$

Let  $u = x^2 + 1$ , giving  $xdx = \frac{1}{2}du$ .

$$x = 0 \Rightarrow u = 1, \quad x = \sqrt{3} \Rightarrow u = 4.$$

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2 + 1}} dx = \frac{1}{2} \int_1^4 u^{-1/2} du = [\sqrt{u}]_1^4 = \boxed{1}.$$

**Two valid routes:** change the limits and stay in  $u$ ; or find the antiderivative, return to  $x$ , then use the original limits. Never use  $x$ -limits with a  $u$ -antiderivative.

#### Other inner functions to recognise

| Integral pattern          | Try          | Differential      |
|---------------------------|--------------|-------------------|
| $\int xe^{x^2} dx$        | $u = x^2$    | $du = 2xdx$       |
| $\int \sin x \cos^4 x dx$ | $u = \cos x$ | $du = -\sin x dx$ |
| $\int (3x - 2)^5 dx$      | $u = 3x - 2$ | $du = 3dx$        |

#### 4 HL substitution: rewrite every remaining factor

At HL, the derivative may not already be present. The substitution must also allow you to rewrite remaining factors of  $x$  and any square roots.

**Syllabus guidance.** For AHL 5.16, examination papers provide a substitution when the integral is not of the SL form  $\int k g'(x) f(g(x)) dx$ . Practise following the given substitution accurately; the examples below show why it helps.

##### Worked example D — a leftover factor of $x$

$$\text{Find } \int x\sqrt{x+2} dx \quad \text{using } u = x + 2.$$

Then  $x = u - 2$  and  $dx = du$ . All three pieces must change:

$$\int x\sqrt{x+2} dx = \int (u-2)u^{1/2} du = \int (u^{3/2} - 2u^{1/2}) du.$$

$$\boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C} \quad (x > -2).$$

##### Worked example E — remove a square root

$$\text{Find } \int \frac{1}{1+\sqrt{x}} dx \quad \text{using } x = u^2, u \geq 0.$$

Then  $\sqrt{x} = u$  and  $dx = 2u du$ . Hence

$$\int \frac{1}{1+\sqrt{x}} dx = \int \frac{2u}{1+u} du = 2 \int \left(1 - \frac{1}{1+u}\right) du.$$

$$\boxed{2\sqrt{x} - 2\ln(1+\sqrt{x}) + C} \quad (x > 0).$$

**Why the differential matters:**  $dx$  becomes  $2u du$ , not  $du$ .

##### Worked example F — trigonometric substitution

$$\text{Find } \int \sqrt{9-x^2} dx \quad \text{using } x = 3 \sin \theta.$$

For  $-3 < x < 3$ , choose  $-\pi/2 < \theta < \pi/2$ , so  $\cos \theta > 0$ . Then

$$dx = 3 \cos \theta d\theta, \quad \sqrt{9-x^2} = 3\sqrt{1-\sin^2 \theta} = 3 \cos \theta.$$

$$\int \sqrt{9-x^2} dx = 9 \int \cos^2 \theta d\theta = \frac{9}{2} \int (1 + \cos 2\theta) d\theta = \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta + C.$$

Since  $\theta = \arcsin(x/3)$  and  $\sin 2\theta = 2(x/3)(\sqrt{9-x^2}/3)$ ,

$$\boxed{\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2} \arcsin\left(\frac{x}{3}\right) + C}.$$

**Sign check:**  $\sqrt{\cos^2 \theta} = |\cos \theta|$ . The chosen interval is what permits us to replace this by  $\cos \theta$ .

## 5 SL substitution practice

For Q7–12, choose your own substitution. Show the transformed integral. All indefinite answers require  $+C$ .

7. Find  $\int 6x(x^2 + 1)^4 dx$ .

8. Find  $\int \frac{3x^2}{x^3 + 4} dx$ . State where your answer applies.

9. Find  $\int 2xe^{x^2} dx$ . Check your answer by differentiation.

10. Find  $\int \sin x \cos^3 x dx$ . Explain the origin of the negative sign in your answer.

11. Evaluate  $\int_0^1 x\sqrt{x^2 + 3} dx$ . Write down the new lower and upper limits explicitly.

12. Find each integral, including the constant factor arising from the substitution.

(a)  $\int \cos(3x + 1) dx$ .

(b)  $\int \frac{1}{2x - 5} dx$ .

### Self-check before moving to HL

1. Can you identify both the inner function and its derivative?
2. Have you accounted for any missing numerical factor or minus sign?
3. Does your transformed integral contain only  $u$  and  $du$ ?
4. Have you returned to  $x$ , or used the correct  $u$ -limits?

**Connection to kinematics.** In Q2 the change from time to velocity uses  $ds = (v/a)dv$ . This is the same central idea: change the variable *and* the differential together.

## 6 HL substitution and connected practice

Use the substitution stated in each question. For indefinite integrals, give an answer in terms of  $x$ .

13. Using  $u = x + 1$ , find  $\int x\sqrt{x+1} \, dx$ , for  $x > -1$ .

14. Using  $x = u^2$ , where  $u > 0$ , find  $\int \frac{1}{\sqrt{x}(1+\sqrt{x})} \, dx$ , for  $x > 0$ .

15. Using  $u = 1 + x^2$ , evaluate  $\int_0^1 \frac{x^3}{\sqrt{1+x^2}} \, dx$ . *Hint: write  $x^3 dx = x^2(x dx)$ .*

16. Using  $x = 2 \sin \theta$ , evaluate  $\int_0^1 \sqrt{4-x^2} \, dx$ . State the new limits and explain the sign of  $\cos \theta$  on the interval used.

17. Using  $x = 3 \tan \theta$ , with  $-\pi/2 < \theta < \pi/2$ , find  $\int \frac{1}{x^2+9} \, dx$ .

18. **Bring the topics together.** A particle moves in the positive direction with  $v(s) = \sqrt{9-s^2}$ , for  $0 \leq s < 3$ . At  $t = 0$  it is at  $s = 0$ . Use SI units.

- Show that its acceleration is  $a = -s$ .
- Set up an integral for the time taken to reach  $s = 3/2$ .
- Evaluate this integral using  $s = 3 \sin \theta$ .
- Find  $s(t)$  and state the time interval corresponding to  $0 \leq s < 3$ .

**Final checks.** Time taken should be positive. Displacement may be positive, negative or zero. Total distance is non-negative. If a particle changes direction, split the distance integral at each sign change of velocity.

## Detailed solutions / Kinematics

### 1 Supplied question: $v(s) = 1/s$

**(a) Choose the correct derivative.** Acceleration is  $dv/dt$ , but the given velocity is a function of  $s$ . Use the chain rule:

$$a = \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds}.$$

Since  $v = s^{-1}$ , we have  $dv/ds = -s^{-2}$ . Hence

$$a = \frac{1}{s} \left( -\frac{1}{s^2} \right) = -\frac{1}{s^3}, \quad \boxed{a(2) = -\frac{1}{8} \text{ m s}^{-2}}.$$

Velocity is positive and acceleration negative, so the particle is slowing down.

**(b) Obtain time from the velocity definition.**

$$v = \frac{ds}{dt} = \frac{1}{s} \implies dt = s ds.$$

Take  $t = 0$  at  $s = 2$ , and let  $T$  be the elapsed time at  $s = 4$ :

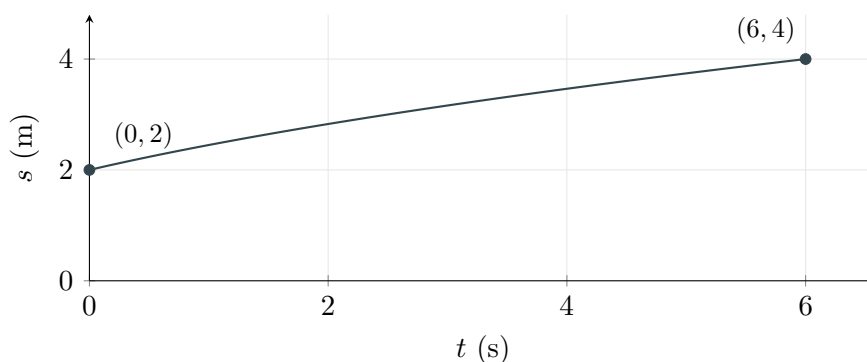
$$\int_0^T dt = \int_2^4 s ds \implies T = \left[ \frac{s^2}{2} \right]_2^4 = 8 - 2 = \boxed{6 \text{ s}}.$$

**Why not  $\int_2^4 v(s) ds$ ?** The relation for displacement is  $ds = v dt$ . Here we want time, so we need  $dt = ds/v$ .

**Independent check.** Integrating  $s ds = dt$  gives  $s^2 = 2t + C$ . The initial condition gives  $C = 4$ , so

$$s(t) = \sqrt{2t + 4}, \quad v(t) = \frac{1}{\sqrt{2t + 4}}.$$

At  $t = 6$ ,  $s = 4$ , as required.



The displacement increases while the gradient decreases: the particle continues forwards at a decreasing speed.

## Detailed solutions / The second supplied question

### 2 Supplied question: $a = -(1 + v)$

Let  $s_0$  be the starting displacement. We know  $v = 1$  initially and  $v = 0$  at first rest, so use velocity as the integration variable.

**Step 1: connect acceleration to displacement.**

$$a = v \frac{dv}{ds} = -(1 + v) \implies ds = -\frac{v}{1 + v} dv.$$

This is valid on the motion before first rest; the integral extends continuously to  $v = 0$ .

**Step 2: use the correct order of limits.**

$$\Delta s = s_{\text{rest}} - s_0 = \int_1^0 -\frac{v}{1 + v} dv = \int_0^1 \frac{v}{1 + v} dv.$$

The decreasing velocity makes the original limits run from 1 to 0; this cancels the negative sign.

**Step 3: simplify the fraction before integrating.**

$$\frac{v}{1 + v} = \frac{(1 + v) - 1}{1 + v} = 1 - \frac{1}{1 + v}.$$

$$\Delta s = \int_0^1 \left(1 - \frac{1}{1 + v}\right) dv = [v - \ln(1 + v)]_0^1 = \boxed{1 - \ln 2 \text{ m}}.$$

Numerically,  $\Delta s \approx 0.307$  m. Velocity remains positive before first rest, so the distance travelled is also  $1 - \ln 2$  metres.

**Explicit substitution version.** In  $\int_0^1 v/(1 + v) dv$ , put  $u = 1 + v$ , so  $v = u - 1$  and  $dv = du$ . The limits become 1 and 2:

$$\Delta s = \int_1^2 \frac{u - 1}{u} du = [u - \ln u]_1^2 = 1 - \ln 2.$$

### Alternative verification: solve for velocity as a function of time

Starting with  $dv/dt = -(1 + v)$  and  $v(0) = 1$ ,

$$\int_1^{v(t)} \frac{1}{1 + w} dw = -\int_0^t d\tau \implies \ln\left(\frac{1 + v}{2}\right) = -t.$$

Thus  $v(t) = 2e^{-t} - 1$ . First rest occurs when  $e^{-t} = 1/2$ , giving  $t = \ln 2$ .

$$\Delta s = \int_0^{\ln 2} (2e^{-t} - 1) dt = [-2e^{-t} - t]_0^{\ln 2} = \boxed{1 - \ln 2 \text{ m}}.$$

**Rest does not mean zero acceleration.** At  $v = 0$ ,  $a = -1 \text{ ms}^{-2}$ . If the model continues, the particle reverses direction after this instant; it does not stay at rest.

## Detailed solutions / Further kinematics

### 3 Same algebra, different independent variables

(a) Here  $v = v(t)$ , so  $a = dv/dt = 6t$ . At  $t = 2$ ,  $a = 12 \text{ m s}^{-2}$ .

(b) Here  $v = v(s)$ , so  $a = v dv/ds = (3s^2 + 2)(6s)$ . At  $s = 2$ ,  $v = 14$  and  $dv/ds = 12$ , giving  $a = 168 \text{ m s}^{-2}$ . In (b), the derivative  $6s$  is change in velocity per unit displacement; multiplying by  $ds/dt = v$  converts it to change per unit time.

### 4 $v(s) = 4/s$

(a)  $a = v dv/ds = (4/s)(-4/s^2) = -16/s^3$ . Thus  $a(2) = -2 \text{ m s}^{-2}$ .

(b) Since  $dt = ds/v = (s/4)ds$ ,

$$T = \int_2^6 \frac{s}{4} ds = \left[ \frac{s^2}{8} \right]_2^6 = \frac{36 - 4}{8} = 4 \text{ s}.$$

(c) Integrate between the initial position and the current position:

$$t = \int_2^{s(t)} \frac{w}{4} dw = \frac{s^2 - 4}{8} \Rightarrow s(t) = \sqrt{8t + 4} \text{ m}, \quad t \geq 0.$$

Choose the positive root because  $s > 0$ .

### 5 $a = -(2 + v)$ and $v(0) = 4$

(a) Using  $a = v dv/ds$ , we obtain

$$\Delta s = \int_4^0 -\frac{v}{2 + v} dv = \int_0^4 \left( 1 - \frac{2}{2 + v} \right) dv.$$

$$\Delta s = [v - 2 \ln(2 + v)]_0^4 = 4 - 2 \ln 6 + 2 \ln 2 = 4 - 2 \ln 3 \text{ m}.$$

(b) From  $a = dv/dt$ ,  $dt = -dv/(2 + v)$ :

$$T = \int_4^0 -\frac{1}{2 + v} dv = [-\ln(2 + v)]_4^0 = \ln 6 - \ln 2 = \ln 3 \text{ s}.$$

(c) Before first rest,  $v$  decreases from 4 to 0 and stays positive. There is no reversal, so distance = displacement =  $4 - 2 \ln 3$  metres.

## Detailed solutions / Direction and distance

6  $v(t) = t^2 - 4t + 3$

(a) Factorise:  $v = (t - 1)(t - 3)$ . The particle is at rest at  $t = 1 \text{ s}, 3 \text{ s}$ . Velocity is positive on  $0 \leq t < 1$ , negative on  $1 < t < 3$ , and positive on  $3 < t \leq 4$ . It reverses at both zeros.

(b) An antiderivative is  $F(t) = t^3/3 - 2t^2 + 3t$ . Thus

$$\Delta s = F(4) - F(0) = \frac{64}{3} - 32 + 12 = \frac{4}{3} \text{ m}.$$

For distance, split at both sign changes and make each contribution positive:

$$D = \int_0^1 v \, dt - \int_1^3 v \, dt + \int_3^4 v \, dt.$$

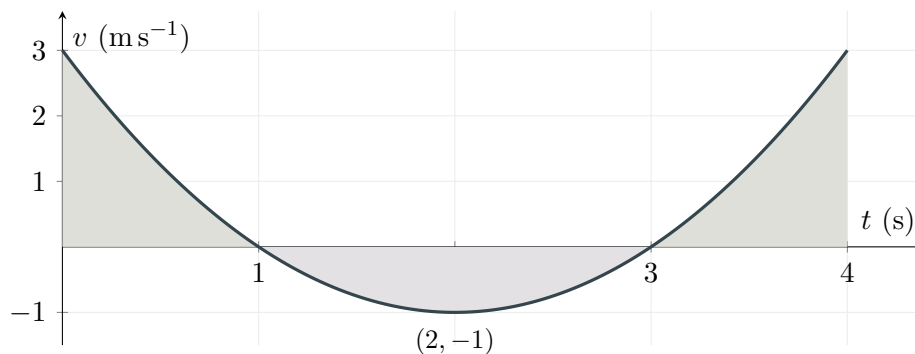
Since  $F(0) = 0$ ,  $F(1) = 4/3$ ,  $F(3) = 0$  and  $F(4) = 4/3$ ,

$$D = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4 \text{ m}.$$

(c) Speed increases when  $v$  and  $a$  have the same sign. Here  $a = dv/dt = 2t - 4$ , which changes sign at  $t = 2$ .

| Interval | (0, 1)     | (1, 2)     | (2, 3)     | (3, 4)     |
|----------|------------|------------|------------|------------|
| $v$      | +          | -          | -          | +          |
| $a$      | -          | -          | +          | +          |
| Speed    | decreasing | increasing | decreasing | increasing |

Therefore speed increases for  $1 < t < 2$  and  $3 < t < 4$ .



**Read the shaded regions:** displacement counts the middle region negatively. Distance counts the magnitude of all three regions. Here each region has area  $4/3$ .

## Detailed solutions / SL substitution

### 7–9 Powers, logarithms and exponentials

7. Let  $u = x^2 + 1$ , so  $du = 2x dx$  and  $6x dx = 3du$ .

$$\int 6x(x^2 + 1)^4 dx = 3 \int u^4 du = \boxed{\frac{3}{5}(x^2 + 1)^5 + C}.$$

8. Let  $u = x^3 + 4$ , so  $du = 3x^2 dx$ .

$$\int \frac{3x^2}{x^3 + 4} dx = \int \frac{1}{u} du = \boxed{\ln |x^3 + 4| + C}.$$

This holds on any interval that does not contain  $x = -\sqrt[3]{4}$ ; constants may differ on the two intervals of the domain.

9. Let  $u = x^2$ , so  $du = 2x dx$ .

$$\int 2xe^{x^2} dx = \int e^u du = \boxed{e^{x^2} + C}.$$

Check:  $d(e^{x^2})/dx = 2xe^{x^2}$ .

### 10–12 Trigonometry, limits and linear inner functions

10. Let  $u = \cos x$ , so  $du = -\sin x dx$ .

$$\int \sin x \cos^3 x dx = - \int u^3 du = \boxed{-\frac{1}{4} \cos^4 x + C}.$$

The negative sign comes from differentiating  $\cos x$ .

11. Let  $u = x^2 + 3$ , giving  $x dx = \frac{1}{2} du$ . The limits are  $x = 0 \Rightarrow u = 3$  and  $x = 1 \Rightarrow u = 4$ .

$$\int_0^1 x \sqrt{x^2 + 3} dx = \frac{1}{2} \int_3^4 u^{1/2} du = \left[ \frac{1}{3} u^{3/2} \right]_3^4 = \boxed{\frac{8 - 3\sqrt{3}}{3}}.$$

12(a). Let  $u = 3x + 1$ , so  $dx = \frac{1}{3} du$ :

$$\int \cos(3x + 1) dx = \frac{1}{3} \int \cos u du = \boxed{\frac{1}{3} \sin(3x + 1) + C}.$$

12(b). Let  $u = 2x - 5$ , so  $dx = \frac{1}{2} du$ :

$$\int \frac{1}{2x - 5} dx = \frac{1}{2} \int \frac{1}{u} du = \boxed{\frac{1}{2} \ln |2x - 5| + C}, \quad x \neq \frac{5}{2}.$$

## Detailed solutions / HL algebraic substitutions

### 13 Rewrite the remaining $x$

Put  $u = x + 1$ . Then  $x = u - 1$ ,  $\sqrt{x+1} = u^{1/2}$  and  $dx = du$ .

$$\begin{aligned}\int x\sqrt{x+1} \, dx &= \int (u-1)u^{1/2} \, du \\ &= \int (u^{3/2} - u^{1/2}) \, du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C.\end{aligned}$$

Thus  $\boxed{\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C}$ .

Check: differentiating gives  $(x+1)^{3/2} - (x+1)^{1/2} = x\sqrt{x+1}$ .

### 14 The differential cancels a radical

Put  $x = u^2$ , with  $u > 0$ , so  $\sqrt{x} = u$  and  $dx = 2u \, du$ .

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})} \, dx = \int \frac{2u}{u(1+u)} \, du = 2 \int \frac{1}{1+u} \, du = 2\ln(1+u) + C.$$

Therefore  $\boxed{2\ln(1+\sqrt{x}) + C}$ , for  $x > 0$ .

Check:  $2(1+\sqrt{x})^{-1}(2\sqrt{x})^{-1} = 1/[\sqrt{x}(1+\sqrt{x})]$ .

### 15 Split the factor $x^3$

Put  $u = 1 + x^2$ . Then  $x^2 = u - 1$  and  $x \, dx = \frac{1}{2}du$ . The limits are  $u = 1$  and  $u = 2$ .

$$\begin{aligned}\int_0^1 \frac{x^3}{\sqrt{1+x^2}} \, dx &= \frac{1}{2} \int_1^2 \frac{u-1}{\sqrt{u}} \, du \\ &= \frac{1}{2} \int_1^2 (u^{1/2} - u^{-1/2}) \, du \\ &= \left[ \frac{1}{3}u^{3/2} - u^{1/2} \right]_1^2 \\ &= \left( \frac{2\sqrt{2}}{3} - \sqrt{2} \right) - \left( \frac{1}{3} - 1 \right) = \boxed{\frac{2-\sqrt{2}}{3}}.\end{aligned}$$

The answer is positive, consistent with the non-negative integrand on  $[0, 1]$ .

## Detailed solutions / HL trigonometric substitutions

### 16 A semicircle integral

Use  $x = 2 \sin \theta$ . The limits become  $x = 0 \Rightarrow \theta = 0$  and  $x = 1 \Rightarrow \theta = \pi/6$ . On this interval  $\cos \theta > 0$ , so

$$\sqrt{4 - x^2} = 2 \cos \theta, \quad dx = 2 \cos \theta d\theta.$$

Therefore

$$\begin{aligned} \int_0^1 \sqrt{4 - x^2} dx &= 4 \int_0^{\pi/6} \cos^2 \theta d\theta \\ &= 2 \int_0^{\pi/6} (1 + \cos 2\theta) d\theta \\ &= [2\theta + \sin 2\theta]_0^{\pi/6} = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}. \end{aligned}$$

### 17 Use $1 + \tan^2 \theta = \sec^2 \theta$

With  $x = 3 \tan \theta$ , we have

$$dx = 3 \sec^2 \theta d\theta, \quad x^2 + 9 = 9(1 + \tan^2 \theta) = 9 \sec^2 \theta.$$

Thus

$$\int \frac{1}{x^2 + 9} dx = \int \frac{3 \sec^2 \theta}{9 \sec^2 \theta} d\theta = \frac{1}{3} \theta + C = \boxed{\frac{1}{3} \arctan\left(\frac{x}{3}\right) + C}.$$

The specified interval makes  $\theta = \arctan(x/3)$  unambiguous.

### What changed from SL to HL?

- In Q13, the substitution also rewrote a factor outside the square root.
- In Q14, the transformed differential supplied a factor that cancelled.
- In Q15,  $x^3 dx$  had to be split into  $x^2(x dx)$ .
- In Q16–17, an identity simplified the integral after substitution.

In every case, the basic rule remained the same: rewrite the entire integral consistently in the new variable.

## Detailed solutions / Connecting both topics

18  $v(s) = \sqrt{9 - s^2}$

(a) **Acceleration.** Since velocity is given in terms of displacement,

$$a = v \frac{dv}{ds}, \quad \frac{dv}{ds} = \frac{1}{2}(9 - s^2)^{-1/2}(-2s) = -\frac{s}{\sqrt{9 - s^2}}.$$

Thus  $a = -s$  for  $0 \leq s < 3$ .

(b) **Time integral.** From  $ds/dt = v$ , we get  $dt = ds/v$ . Hence

$$T = \int_0^{3/2} \frac{1}{\sqrt{9 - s^2}} ds.$$

(c) **Substitution.** Put  $s = 3 \sin \theta$ , so  $ds = 3 \cos \theta d\theta$ . The limits are 0 and  $\pi/6$ . On this interval,  $\cos \theta > 0$ , so

$$T = \int_0^{\pi/6} \frac{3 \cos \theta}{3 \cos \theta} d\theta = \left[ \frac{\pi}{6} s \right].$$

(d) **Displacement as a function of time.** Integrate to a general displacement  $s$ :

$$t = \int_0^s \frac{1}{\sqrt{9 - w^2}} dw = \arcsin\left(\frac{s}{3}\right).$$

Therefore  $s(t) = 3 \sin t$  m,  $0 \leq t < \pi/2$ . On this interval  $\cos t > 0$ , so  $v(t) = 3 \cos t = \sqrt{9 - s(t)^2}$  as required.

**Domain matters.** The given model specifies  $s < 3$ . The limiting position  $s = 3$  is approached as  $t \rightarrow \pi/2$  from below. Do not extend the positive-square-root velocity formula past this interval without a new analysis.

## Closing checklist

**Kinematics** Is the input  $t$ ,  $s$  or  $v$ ? Do I need  $a = dv/dt$  or  $a = v dv/ds$ ? Am I finding displacement, distance or time?

**Substitution** Did I transform the differential? Have all old-variable factors disappeared? Are the limits and signs correct? Can I check by differentiating?

**Source and scope note.** Syllabus references checked against the supplied *Mathematics: analysis and approaches guide*, printed pp.65 and 68. Questions 1–2 reproduce the mathematical content of the supplied image; Q3–18 are original practice questions, not official IB examination questions. This resource focuses on the substitution component of AHL 5.16; integration by parts is outside its scope.