

The Discriminant Δ

IB Mathematics: Analysis and Approaches SL & HL – SL 2.7

50 questions on $\Delta = b^2 - 4ac$, the nature of roots, parameter problems, intersections and tangency

Learning goal

Use the discriminant to classify roots and solve increasingly demanding quadratic, parameter, graph-intersection and tangency problems.

Syllabus link

IB Mathematics AA SL/HL: **SL 2.7** – quadratic equations and inequalities; factorization, completing the square and the quadratic formula; $\Delta = b^2 - 4ac$ and the nature of the roots.

Structure

Section 1: 15 short-response questions; Section 2: 15 extended-response questions; Section 3: 20 original IB-style Paper 1 questions.

Calculator guidance

Designed as non-calculator practice. Give exact values unless a question specifically says otherwise.

Discriminant reference

For $ax^2 + bx + c = 0$ with $a \neq 0$,

$$\Delta = b^2 - 4ac.$$

$\Delta > 0$: two distinct real roots

$\Delta = 0$: two equal real roots

$\Delta < 0$: no real roots.

Instructions

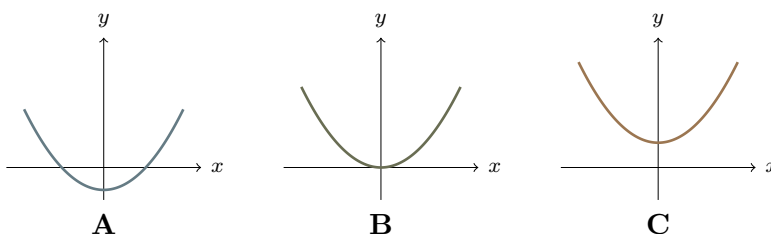
Show enough working to make your use of the discriminant clear. In parameter questions, remember to check whether the coefficient of x^2 can become zero; in that case the equation is no longer quadratic. When a question is about a graph, connect the sign of Δ to the number of intersections with the relevant line or axis.

Section 1: Short response

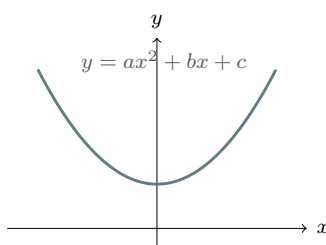
Questions 1–15

- For $x^2 - 6x + 5 = 0$, calculate Δ and state the nature of the roots.
- For $2x^2 + 4x + 5 = 0$, calculate Δ and state the nature of the roots.
- For $9x^2 - 12x + 4 = 0$, calculate Δ and state the nature of the roots.
- For $x^2 + 7x + 10 = 0$, calculate Δ and hence solve the equation.

5. For $3x^2 - 5x - 2 = 0$, calculate Δ and hence solve the equation.
6. Use the discriminant to explain why $x^2 + 4x + 7 = 0$ has no real solution.
7. Find the values of k for which $x^2 + kx + 9 = 0$ has two equal real roots.
8. Find the values of m for which $x^2 - 4x + m = 0$ has two distinct real roots.
9. Find the values of p for which $2x^2 + px + 8 = 0$ has no real roots.
10. Find the value of k for which $3x^2 + 2x + k = 0$ has two equal real roots.
11. State the number of x -intercepts of the graph $y = x^2 + 2x + 5$. Justify your answer using Δ .
12. The three sketches below show the graphs of three quadratic functions. For each graph, state whether the discriminant is positive, zero or negative.



13. The graph $y = 2x^2 - 8x + 8$ touches the x -axis. Use the discriminant to confirm this and find the point of contact.
14. The graphs $y = x^2$ and $y = 4x - c$ intersect at two distinct points. Find the possible values of c . State what happens when $c = 4$.
15. The sketch below shows the graph of $y = ax^2 + bx + c$. State the sign of a and the sign of Δ .



Section 2: Extended response

Questions 16–30

16. Consider the equation

$$3kx^2 + 2x + k = 0.$$

For $k \neq 0$, determine the values of k for which the equation has (a) two distinct real roots, (b) two equal real roots, and (c) no real roots. State separately what happens when $k = 0$.

17. Determine the values of k for which $kx^2 + 6x + k = 0$ is a quadratic equation with two distinct real roots.

18. Find all values of k for which

$$x^2 + (k - 2)x + k = 0$$

has two equal real roots.

19. Determine the values of k for which

$$x^2 - 2(k + 1)x + k^2 + 1 = 0$$

has no real roots.

20. Determine the values of k for which

$$(k + 1)x^2 - 4x + 1 = 0$$

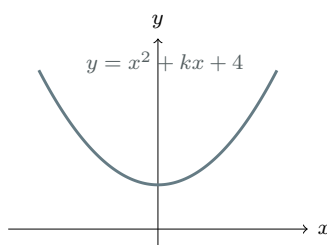
is a quadratic equation with two distinct real roots.

21. The line $y = 2x + k$ is tangent to the parabola $y = x^2 - 4x + 7$. Find k and the coordinates of the point of tangency.

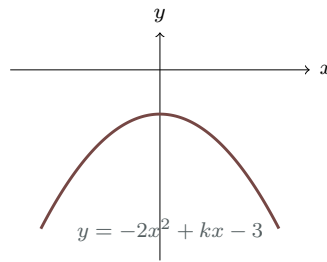
22. The line $y = mx + 1$ is tangent to the parabola $y = x^2 - 2x + 3$. Find the possible values of m .

23. Show that the line $y = px + 4$ intersects the parabola $y = 3x^2 + 1$ at two distinct points for every real value of p .

24. The sketch below shows the graph of $y = x^2 + kx + 4$ opening upward and lying entirely above the x -axis. Use the discriminant to find the possible values of k .



- 25.** The sketch below shows the graph of $y = -2x^2 + kx - 3$ opening downward and lying entirely below the x -axis. Use the discriminant to find the possible values of k .



- 26.** Find the values of c for which

$$x^2 - 6x + c \geq 0$$

for every real value of x .

- 27.** Find the values of k for which

$$2x^2 + kx + 8 > 0$$

for every real value of x .

- 28.** Find the values of k for which

$$x^2 + (k - 1)x + (k + 2) = 0$$

has no real roots.

- 29.** Determine the values of p for which

$$x^2 - (p + 2)x + 2p = 0$$

has two distinct real roots.

- 30.** The equation

$$x^2 + 2kx + k + 5 = 0$$

has two distinct positive real roots. Determine the possible values of k .

Section 3: IB-style Paper 1

Questions 31–50

- 31.** Consider the function $f(x) = x^2 - 6x + k$, where $k \in \mathbb{R}$. The graph of f touches the x -axis.

- (a) Find the value of k . [2 marks]
- (b) Hence find the coordinates of the point where the graph touches the x -axis. [2 marks]



32. Consider the equation $3x^2 + mx + 12 = 0$, where m is a positive integer. The equation has no real roots. Find the greatest possible value of m . [3 marks]

33. The equation

$$(k - 1)x^2 + 4x + 2 = 0$$

has two equal real roots.

(a) Find k . [2 marks]

(b) Hence find the repeated root. [2 marks]

34. The line L has equation $y = 2x + c$ and is tangent to the parabola $y = x^2 - 2x + 5$.

(a) Find c . [3 marks]

(b) Find the coordinates of the point of tangency. [2 marks]

35. The line $y = mx - 1$ intersects the parabola $y = x^2 + 3$ at two distinct points. Find the possible values of m . [3 marks]

36. Consider $q(x) = x^2 + (a - 3)x + a$.

(a) Find the values of a for which $q(x) = 0$ has two equal real roots. [3 marks]

(b) For each value of a , find the repeated root. [2 marks]

37. The function P is defined by

$$P(x) = -x^2 + 6x - k, \quad x \in \mathbb{R}.$$

Find the values of k for which $P(x) < 0$ for every real value of x . [4 marks]

38. The line $y = kx + 4$ and the parabola $y = 2x^2 + 3x + 1$ are given. Show that there is no real value of k for which the line is tangent to the parabola. [3 marks]

39. The graph of $y = x^2 + kx + 9$ has exactly one x -intercept.

(a) Find the possible values of k . [2 marks]

(b) Find the x -intercept for each value of k . [3 marks]

40. Determine the values of k for which

$$x^2 + (k + 1)x + 2k > 0$$

for all $x \in \mathbb{R}$. [4 marks]



41. The equation

$$x^2 - 2kx + k + 6 = 0$$

has two distinct positive real roots. Find the possible values of k . [5 marks]

42. The equation

$$x^2 + (k - 1)x + k - 2 = 0$$

has two distinct negative real roots. Find the possible values of k . [5 marks]

43. The equation

$$x^2 + (k - 4)x - 2k = 0$$

has one positive root and one negative root. Find the possible values of k . [3 marks]

44. The point $A(0, 3)$ lies outside the parabola $y = x^2 - 4x + 7$. A line through A has equation $y = mx + 3$.

(a) Find the two values of m for which the line is tangent to the parabola. [3 marks]

(b) Find the coordinates of both points of tangency. [3 marks]

45. A line through the point $(0, 10)$ has equation $y = mx + 10$. Show that no such line is tangent to the parabola $y = x^2 + 2x + 5$. [3 marks]

46. Find the values of k for which

$$x^2 + kx + k^2 - 4 = 0$$

has at least one real root. [3 marks]

47. The roots of $x^2 - 6x + c = 0$ are distinct and differ by 4. Find c and hence find the two roots. [5 marks]

48. The two real roots of

$$2x^2 + kx + 2 = 0$$

differ by 3. Find the possible values of k . [4 marks]

49. The line $y = x + k$ intersects the parabola $y = -x^2 + 5$.

(a) Find the values of k for which there are two distinct points of intersection. [2 marks]

(b) Find the value of k for which the line is tangent to the parabola. [1 marks]

(c) Find the values of k for which there are no points of intersection. [1 marks]



50. Consider the family of equations

$$(k+1)x^2 - 2(k-2)x + k + 5 = 0, \quad k \in \mathbb{R}.$$

Classify all real values of k according to whether the equation has two distinct real roots, two equal real roots, no real roots, or is not quadratic. [6 marks]

Answer key and solution outlines

Questions 1–50

Section 1: Short response

1. $\Delta = (-6)^2 - 4(1)(5) = 16 > 0$: two distinct real roots.
2. $\Delta = 4^2 - 4(2)(5) = -24 < 0$: no real roots.
3. $\Delta = (-12)^2 - 4(9)(4) = 0$: two equal real roots.
4. $\Delta = 49 - 40 = 9$. Roots $x = \frac{-7 \pm 3}{2}$, so $x = -5, -2$.
5. $\Delta = 25 + 24 = 49$. Roots $x = \frac{5 \pm 7}{6}$, so $x = 2, -\frac{1}{3}$.
6. $\Delta = 4^2 - 4(1)(7) = -12 < 0$, so there are no real solutions.
7. $k^2 - 36 = 0 \Rightarrow k = \pm 6$.
8. $16 - 4m > 0 \Rightarrow m < 4$.
9. $p^2 - 64 < 0 \Rightarrow -8 < p < 8$.
10. $4 - 12k = 0 \Rightarrow k = \frac{1}{3}$.
11. $\Delta = 2^2 - 4(1)(5) = -16 < 0$, so there are no x -intercepts.
12. Graph A crosses the x -axis twice, so $\Delta > 0$. Graph B touches the x -axis once, so $\Delta = 0$. Graph C does not meet the x -axis, so $\Delta < 0$.
13. $\Delta = (-8)^2 - 4(2)(8) = 0$. The repeated root is $x = \frac{8}{4} = 2$, so the point is $(2, 0)$.
14. $x^2 = 4x - c \Rightarrow x^2 - 4x + c = 0$. Two distinct intersections require $16 - 4c > 0$, hence $c < 4$. When $c = 4$, $\Delta = 0$, so the line is tangent to the parabola and the graphs meet at one repeated point.
15. Since the parabola opens upward, $a > 0$. Since it lies entirely above the x -axis, it has no real roots, so $\Delta < 0$.

Section 2: Extended response

16. For $k \neq 0$, $\Delta = 4 - 12k^2$. Thus two distinct roots when $|k| < \frac{1}{\sqrt{3}}$, excluding $k = 0$; equal roots when $k = \pm \frac{1}{\sqrt{3}}$; no real roots when $|k| > \frac{1}{\sqrt{3}}$. At $k = 0$ the equation becomes $2x = 0$, which is linear with one real solution $x = 0$.
17. Need $k \neq 0$ and $\Delta = 36 - 4k^2 > 0$. Hence $-3 < k < 3$, $k \neq 0$.
18. $(k - 2)^2 - 4k = 0 \Rightarrow k^2 - 8k + 4 = 0$, so $k = 4 \pm 2\sqrt{3}$.
19. $\Delta = 4(k + 1)^2 - 4(k^2 + 1) = 8k$. No real roots require $8k < 0$, so $k < 0$.
20. Need $k \neq -1$ and $\Delta = 16 - 4(k + 1) = 12 - 4k > 0$. Thus $k < 3$, $k \neq -1$.
21. Equating gives $x^2 - 6x + (7 - k) = 0$. Tangency: $36 - 4(7 - k) = 0 \Rightarrow k = -2$. Repeated root $x = 3$, and $y = 2(3) - 2 = 4$. Point $(3, 4)$.
22. $x^2 - (m + 2)x + 2 = 0$. Tangency gives $(m + 2)^2 - 8 = 0$, so $m = -2 \pm 2\sqrt{2}$.
23. $3x^2 - px - 3 = 0$ has $\Delta = p^2 + 36 > 0$ for every $p \in \mathbb{R}$, so there are always two distinct intersections.
24. Since the leading coefficient is positive, the graph lies entirely above the axis exactly when there are no real roots: $k^2 - 16 < 0$. Hence $-4 < k < 4$.
25. Since the leading coefficient is negative, the graph lies entirely below the axis exactly when $\Delta < 0$: $k^2 - 24 < 0$. Hence $-2\sqrt{6} < k < 2\sqrt{6}$.
26. A monic quadratic is non-negative for all x when $\Delta \leq 0$: $36 - 4c \leq 0$, so $c \geq 9$.
27. Strict positivity requires $\Delta < 0$: $k^2 - 64 < 0$, hence $-8 < k < 8$.
28. $\Delta = (k - 1)^2 - 4(k + 2) = k^2 - 6k - 7 = (k - 7)(k + 1)$. No roots require $\Delta < 0$, hence $-1 < k < 7$.
29. $\Delta = (p + 2)^2 - 8p = (p - 2)^2$. Two distinct roots require $(p - 2)^2 > 0$, so $p \neq 2$.
30. For two distinct positive roots: sum $= -2k > 0 \Rightarrow k < 0$; product $= k + 5 > 0 \Rightarrow k > -5$; and $\Delta = 4(k^2 - k - 5) > 0$. Thus $k < \frac{1 - \sqrt{21}}{2}$ or $k > \frac{1 + \sqrt{21}}{2}$. Combining gives

$$-5 < k < \frac{1 - \sqrt{21}}{2}.$$

Section 3: IB-style Paper 1 – Questions 31–40

31. $\Delta = 36 - 4k = 0 \Rightarrow k = 9$. Then $f(x) = (x - 3)^2$, so the contact point is $(3, 0)$.
32. No real roots: $m^2 - 144 < 0 \Rightarrow 0 < m < 12$. Since m is a positive integer, the greatest value is 11.
33. $\Delta = 16 - 8(k - 1) = 24 - 8k = 0 \Rightarrow k = 3$. Then $2x^2 + 4x + 2 = 0 \Rightarrow (x + 1)^2 = 0$, so $x = -1$.
34. $x^2 - 2x + 5 = 2x + c \Rightarrow x^2 - 4x + (5 - c) = 0$. Tangency gives $16 - 4(5 - c) = 0 \Rightarrow c = 1$. The repeated root is $x = 2$ and $y = 5$, so the point is $(2, 5)$.
35. $x^2 - mx + 4 = 0$. Two distinct intersections require $m^2 - 16 > 0$, so $m < -4$ or $m > 4$.
36. $\Delta = (a - 3)^2 - 4a = a^2 - 10a + 9 = (a - 1)(a - 9)$. Thus $a = 1$ or 9 . For $a = 1$, $(x - 1)^2 = 0$, root 1. For $a = 9$, $(x + 3)^2 = 0$, root -3 .
37. The parabola opens downward. For $P(x) < 0$ for every real x , it must have no real zeros: $\Delta = 36 - 4k < 0 \Rightarrow k > 9$.
38. Intersections satisfy $2x^2 + (3 - k)x - 3 = 0$. Its discriminant is $(3 - k)^2 + 24 > 0$ for every real k . Therefore there are always two intersections and never a tangent.
39. Exactly one intercept means $k^2 - 36 = 0$, so $k = \pm 6$. If $k = 6$, $(x + 3)^2 = 0$ gives $(-3, 0)$. If $k = -6$, $(x - 3)^2 = 0$ gives $(3, 0)$.
40. Since the coefficient of x^2 is positive, strict positivity for all x requires $\Delta < 0$:

$$(k + 1)^2 - 8k = k^2 - 6k + 1 < 0.$$

The boundary values are $k = 3 \pm 2\sqrt{2}$, so

$$3 - 2\sqrt{2} < k < 3 + 2\sqrt{2}.$$

Section 3: IB-style Paper 1 – Questions 41–50

41. Distinct real roots require $\Delta = 4(k^2 - k - 6) = 4(k - 3)(k + 2) > 0$, so $k < -2$ or $k > 3$. For both roots positive, sum $2k > 0$ and product $k + 6 > 0$. Combining gives $k > 3$.
42. Distinct real roots require $\Delta = (k - 3)^2 > 0$, so $k \neq 3$. Both roots negative require sum $1 - k < 0 \Rightarrow k > 1$ and product $k - 2 > 0 \Rightarrow k > 2$. Hence $k > 2$, $k \neq 3$.
43. Opposite signs require the product of the roots to be negative. Product $= -2k < 0 \Rightarrow k > 0$. Also $\Delta = (k - 4)^2 + 8k = k^2 + 16 > 0$, so the roots are indeed real and distinct. Thus $k > 0$.
44. Equating gives $x^2 - (m + 4)x + 4 = 0$. Tangency requires $(m + 4)^2 - 16 = 0$, so $m = 0$ or $m = -8$. For $m = 0$, repeated root $x = 2$, giving $(2, 3)$. For $m = -8$, repeated root $x = -2$, giving $(-2, 19)$.
45. Equating gives $x^2 + (2 - m)x - 5 = 0$. Its discriminant is $(2 - m)^2 + 20 > 0$ for every real m , so there are always two intersections and no tangent.
46. At least one real root requires $\Delta \geq 0$:

$$k^2 - 4(k^2 - 4) = 16 - 3k^2 \geq 0.$$

Thus $k^2 \leq \frac{16}{3}$, so

$$-\frac{4}{\sqrt{3}} \leq k \leq \frac{4}{\sqrt{3}}.$$

47. For a monic quadratic, the difference between the roots is $\sqrt{\Delta}$. Thus $\sqrt{36 - 4c} = 4$, so $36 - 4c = 16$ and $c = 5$. Then $x^2 - 6x + 5 = 0$, giving roots 1 and 5.

48. For $ax^2 + bx + c = 0$, the root difference is $\frac{\sqrt{\Delta}}{|a|}$. Hence

$$\frac{\sqrt{k^2 - 16}}{2} = 3 \Rightarrow k^2 - 16 = 36 \Rightarrow k^2 = 52.$$

Therefore $k = \pm 2\sqrt{13}$.

49. Intersections satisfy $x^2 + x + (k - 5) = 0$, with $\Delta = 1 - 4(k - 5) = 21 - 4k$. Hence: two intersections when $k < \frac{21}{4}$; tangent when $k = \frac{21}{4}$; no intersections when $k > \frac{21}{4}$.

50. The discriminant is

$$[-2(k - 2)]^2 - 4(k + 1)(k + 5) = -40k - 4.$$

For $k \neq -1$: two distinct real roots when $-40k - 4 > 0$, i.e. $k < -\frac{1}{10}$, excluding $k = -1$; two equal roots when $k = -\frac{1}{10}$; no real roots when $k > -\frac{1}{10}$. At $k = -1$, the x^2 term vanishes and the equation becomes $6x + 4 = 0$, so it is linear, not quadratic.